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MATHEMATICS MAGAZINE

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VOLUME 42 • NOVEMBER 1969 • NUMBER 5

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MATHEMATICS MAGAZINE

STEPHEN A. JENNINGS, *EDITOR*

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ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214.

The MATHEMATICS MAGAZINE is published by the Mathematical Association of America at Washington, D. C., bi-monthly except July–August. Ordinary subscriptions are: 1 year $4.00. Members of the Mathematical Association of America and of Mu Alpha Theta may subscribe at the special rate of 2 years for $6.00. Single copies are 80¢.

Second class postage paid at Washington, D. C. and additional mailing offices.

CONJUGATE DIAMETERS AND THE SPECIAL THEORY OF RELATIVITY

ROBERT C. WREDE, San Jose State College

Many physical problems and theories are investigated and interpreted through geometric models. Even though they have been downgraded in the present mathematical curriculum the conic sections provide many examples. An ellipse is the model of a planetary orbit, a parabola symbolizes a section of a parabolic reflector, and a hyperbola represents the path of an elementary particle when it is deflected by a nucleus of an atom. Beyond their employment in the construction of models, the conic sections illustrate the importance of mathematics as an intellectual discipline because they were a part of the mathematical literature long before a need for them arose.

This article concerns an application of the theory of *conjugate diameters*. This theory no longer is found in most elementary developments of analytic geometry, yet it has a certain mathematical beauty and it has applications in differential geometry and the special theory of relativity. Our interest is in the use of conjugate diameters to construct a geometric model for the special theory of relativity.

Ellipses and hyperbolas are called central conics because they have a center of symmetry. A *diameter* of a central conic is a line segment through the center of symmetry with end points on the conic. (See Figure 1.)

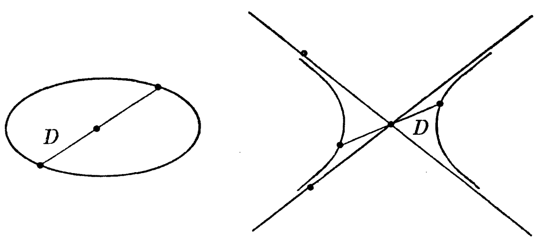

FIG. 1.

Let us concentrate upon hyperbolas for it is their application we wish to consider. The hyperbolas,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1,$$

are called conjugate hyperbolas. (See Figure 2.)

To each diameter associated with conjugate hyperbolas there is another such that the pair satisfy certain properties. These properties are subsequently listed. Their proofs, which depend on simple ideas from analytic geometry and the employment of derivatives to represent slopes, will not be given.

(1) The tangent lines to the hyperbola at the end points of a diameter are parallel.

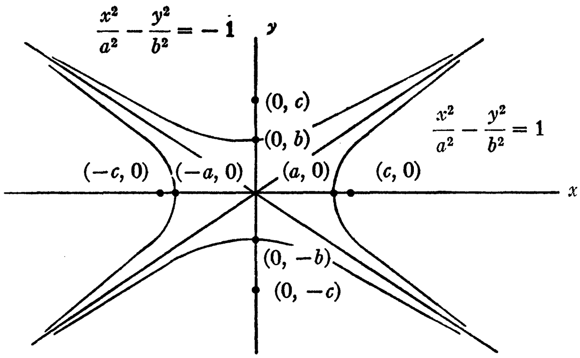

FIG. 2.

(2) Let D_1 and D_2 be diameters of conjugate hyperbolas. If D_2 is parallel to the tangent line at the end point of D_1, then D_1 is parallel to the tangent line at the end point of D_2. (See Figure 3.)

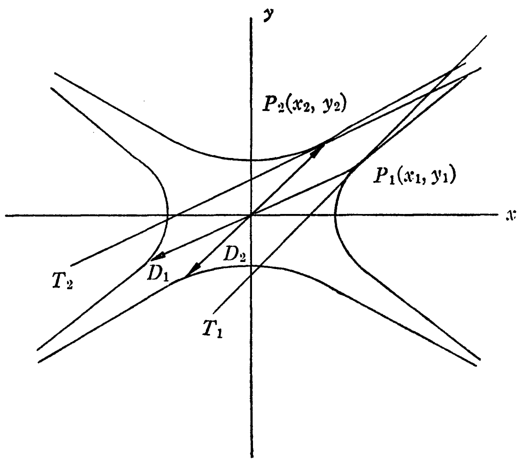

FIG. 3.

Diameters D_1 and D_2 that satisfy property (2) are called *conjugate diameters*. Conjugate diameters can be characterized algebraically. If $P_3(x_3, y_3)$ and $P_4(x_4, y_4)$ are points on the respective conjugate diameters D_1 and D_2, then

$$\frac{x_3x_4}{a^2} - \frac{y_3y_4}{b^2} = 0.$$

The proof of this statement also is simple. It will be omitted. Observe that the common asymptotes of a pair of conjugate hyperbolas satisfy the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0;$$

therefore, for $P_1(x_1, y_1)$ on an asymptote,

$$\frac{x_1 x_1}{a^2} - \frac{y_1 y_1}{b^2} = 0.$$

Thus, asymptotes are self conjugate lines.

Now we are prepared to demonstrate the application of conjugate diameters to the special theory of relativity. The introduction of the theory by Albert Einstein in 1905 was a landmark in physical and, indeed, philosophic thought. As Newton's accomplishments culminated the knowledge of many centuries, so Einstein's synthesized known information from mechanics and electromagnetic theory in a new and powerful manner. Einstein's original paper, *On the electrodynamics of moving bodies*, emphasized physical concepts. In 1908 H. Minkowski introduced a four dimensional space-time model in which three dimensional motions were pictured as paths, that is, because time constituted a space dimension in this model the idea of changing position with variation of time was not included. The Minkowski model demonstrated rather sophisticated philosophic ideas with the elementary tools of analytic geometry that were previously mentioned. Our presentation is based on the model although it varies, particularly in the employment of hyperbolic functions.

The galilean transformations $\bar{x}^j = x^j - v^j t$, $\bar{t} = t$, $j = 1, 2, 3$, are fundamental to classical mechanics. The coordinates x^j and $\bar{x}^j$ are rectangular cartesian ones in euclidean frames of reference 0 and $\bar{0}$, respectively. The frames are in uniform rectilinear motion with respect to one another. The theory of classical mechanics requires that the basic laws, for example Newton's second law of motion, be invariant with respect to the galilean transformations. Also, the transformations illustrate the universal character of time and the parallelogram composition of velocities associated with classical mechanics.

Various prerelativistic experiments indicated inconsistencies between classical mechanics and electromagnetic theory. In particular, the Michelson-Morley experiments found no composition of the earth's velocity with the velocity of light. This fact motivated the following postulate of the special theory of relativity: "The velocity of light, *in vacuo*, is the same in every inertial frame." An *inertial frame* is one in which intrinsic special relations are euclidean and free particles obey Newton's first law when referred to the time measurements universal to the frame. Inertial systems constitute a class of reference frames in uniform rectilinear motion with respect to one another. A group of transformations, called *Lorentz transformations*, results from this postulate, along with the assumption of the linearity of the group. If light is emitted from a point source O(0, 0, 0) it expands spherically according to the formula $c^2t^2 = x^2 + y^2 + z^2$. Let $x = y = 0$ and $c = 1$, then $-z^2 + t^2 = 0$. The postulate requires the invariance of this form. The Lorentz transformations result as linear ones which meet this requirement, that is, $-z^2 + t^2 = -\bar{z}^2 + \bar{t}^2$. If two of the space dimensions are suppressed the four dimensional model reduces to a two dimensional one in which the transformations have the form

$$\bar{z} = z \cosh \psi - t \sinh \psi,$$
$$\bar{t} = - z \sinh \psi + t \cosh \psi,$$

where units have been chosen such that the velocity of light is $c = 1$. The Lorentz transformation equations look much like equations of plane rotations. The fundamental difference in form is that hyperbolic functions appear in them rather than trigonometric ones. The basic invariant of the Lorentz transformations is not z^2+t^2 but rather $-z^2+t^2$. This suggests a new geometry; one in which the euclidean concept of distance $d^2=z^2+t^2$, patterned after the Pythagorean theorem, is replaced by an interval measure

$$I^2 = - z^2 + t^2.$$

The euclidean distance $d^2=z^2+t^2$ is related to the geometry of a circle; the interval measure I^2 is associated with the geometry of conjugate hyperbolas whose asymptotes satisfy the equation $-z^2+t^2=0$. (See Figure 4a.)

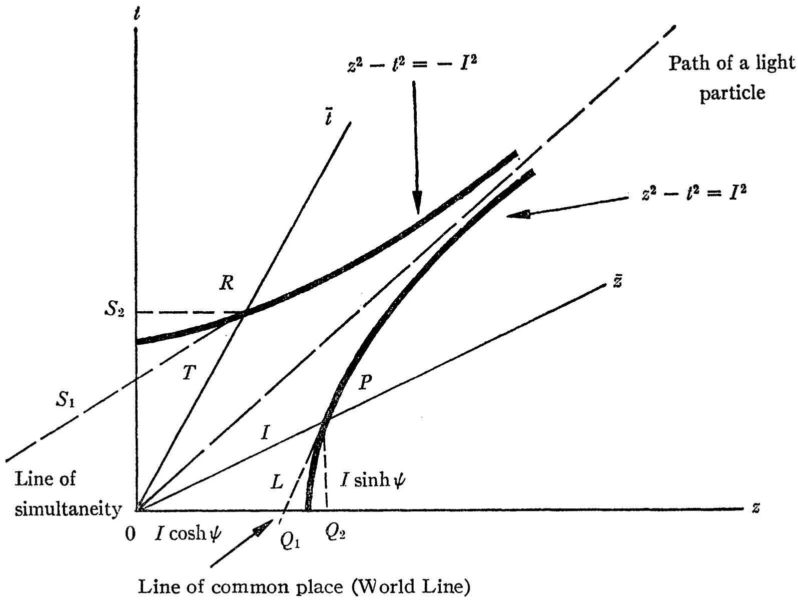

FIG. 4a.

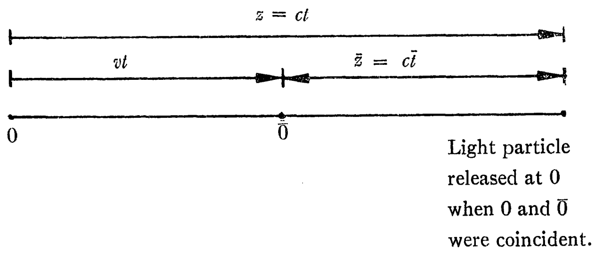

FIG. 4b.

In this new geometry (Minkowskian geometry) the euclidean concept of orthogonality is replaced by one consistent with the Minkowski metric form $I^2 = -z^2 + t^2$. In particular, in the barred system orthogonal directions $[\overline{A}_1, \overline{A}_2]$ and $[\overline{B}_1, \overline{B}_2]$ satisfy the relation

$$- \overline{A}_1\overline{B}_1 + \overline{A}_2\overline{B}_2 = 0.$$

Upon examination of this relation we find that Minkowskian orthogonality is represented by euclidean conjugacy. Beyond this, observe that the Lorentz transformations are affine transformations, therefore parallelism is preserved. Thus, Minkowskian parallelism can be represented by euclidean parallelism. These facts and some further information derived from the Lorentz transformations permit the construction and interpretation of the geometric model for special relativity illustrated in Figure 4.

According to the special theory of relativity, it is not meaningful to say that one system is fixed and another is in motion. The concept of motion is relative. However, to construct a geometric model the point of view of a specific system must be assumed. In Figure 4a we examine the situation from the eyes of an observer in the O system which is referred to rectangular cartesian coordinates z, t. The $\bar{z}$ and $\bar{t}$ axes of the $\bar{\text{O}}$ system also are orthogonal from the standpoint of someone in that system. But in our model (oriented to the O system) they appear as conjugate lines. A particle fixed at the origin of the $\bar{\text{O}}$ system, that is $\bar{z}=0$, has the same velocity as the $\bar{\text{O}}$ system with respect to 0. From the Lorentz equation

$$0 = \bar{z} = z \cosh \psi - t \sinh \psi$$

it follows that this velocity is

$$v = \frac{z}{t} = \frac{\sinh \psi}{\cosh \psi} = \tanh \psi.$$

The velocity v increases as the sinh ψ increases and the cosh ψ decreases. Thus, the velocity v increases as the conjugate axes $\bar{z}$ and $\bar{t}$ come closer together. Since $|\tanh \psi| < 1$, the velocity of light is the limiting velocity of the theory, unattainable for any material particle. The asymptote of the conjugate hyperbolas pictured in Figure 4, that is, the line upon which the conjugate diameters $\bar{z}$ and $\bar{t}$ close as $v \to 1$, represents the path of a light particle (which is not considered to be a material particle).

Also, the model illustrates that two concepts firmly entrenched in man's intuition are not consistent with the special theory of relativity. In Figure 4 the line L is tangent to the hyperbola at P where P also is on the $\bar{z}$ axis. Since the $\bar{t}$ axis is conjugate to the $\bar{z}$ one, L is parallel to the $\bar{t}$ axis. This means that all points on L have the same $\bar{z}$ coordinate and, therefore, correspond to events happening at the same place as observed from the $\bar{\text{O}}$ system. In particular, Q_1 has the same $\bar{z}$ space coordinate as P. For the O observer the event that happens at the same place as P is Q_2, rather than Q_1. This is because the line Q_2P is parallel to the t axis and all points on it have the same space coordinate z. Thus, the concept of the *same place* is a relative one. Our intuition can be adjusted to accept this fact.

We can think of a planet and another celestial body moving away from it at a constant speed which is near the velocity of light. While an observer on the celestial body considers that he is in the same place at all times the planetary observer does not. The corresponding ideas associated with time are more difficult to rationalize.

By examination of the *line of simultaneity* T and the points R, S_1, and S_2 we find that events simultaneous in the $\bar{O}$ system are not simultaneous in the O system, and conversely. This is one of the revolutionary aspects of the special theory of relativity. Its consequences have been thoroughly established by physicists in their study of elementary particles in motion at speeds near that of light.

These are the principal ideas of the special theory of relativity that are related to conjugate diameters. Other information can be extracted from Figure 4 and of course there is much more to be said about the theory itself.

PRIME PRIMES

JOHN E. WALSTROM and MURRAY BERG, San Franciso, California

Consider the prime number 73,939,133. By successively dropping the rightmost digit, the remaining numbers are also primes. Such numbers may be designated as prime primes. The successive prime primes which can be obtained from the above number are 7,393,913; 739,391; 73,939, etc.

Prime primes are relatively simple to obtain and relatively few in number. Beginning with the single digit primes, 2, 3, 5, and 7, and attaching a rightmost digit of 1, 3, 7, or 9, results in all possible numbers consisting of two digits, which may be prime primes. The possible numbers can be verified in a prime number table or can be tested by brute force of a simple computer program. This initial stage results in the prime primes 23, 29, 31, etc. When trying to extend this list from eight digits to nine digits, we find there are no 9-digit prime primes. Set out below is a complete list of prime primes. Prime primes which can be derived by successive elimination of rightmost digits, are not included in this table (e.g., 31 can be derived from 317).

53	3,797	73,331	7,393,931
	7,331	373,393	7,393,933
317	23,333	593,993	23,399,339
599	23,339	719,333	29,399,999
797	31,193	739,397	37,337,999
2,393	31,379	739,399	59,393,339
3,793	37,397	2,399,333	73,939,133.

METHODS OF SOLUTION OF THE RICCATI DIFFERENTIAL EQUATION

D. ROBERT HAAHEIM AND F. MAX STEIN

1. Introduction. There have been several papers recently which have presented methods for the solution of the Riccati equation under certain conditions, (see the references at the end of the paper). Riccati himself was concerned with the solution of the equation

$$y' + by^2 = cx^m, \tag{1}$$

see [**1**]. Today, the equation bearing Riccati's name is written in the more general form (and the form we shall use in this paper),

$$y' + Py + Qy^2 = R, \tag{2}$$

where P, Q, and R are functions of the real independent variable x.

It is the purpose of this paper to present some of the more important methods regarding the solvability of (2) by elementary means, and to present some new methods in this area. Specifically, some fairly easy tests are presented for determining whether or not a given Riccati equation can be solved by a particular elementary method.

The book by Murphy [**2**] contains a number of methods which can be applied to certain Riccati equations. Some of these are listed in tabular form in Section 6 of this paper, along with other recently discussed methods and new methods presented here for the first time.

Since a number of transformations exist for changing a Riccati equation to a second order linear homogeneous equation (and vice versa), the methods of this paper have analogs which can be applied to the solution of such second order equations.

2. Current methods. Under certain conditions on the coefficients, the equation in (2) can be transformed into a Riccati equation in which the variables are separable. Rao [**3**] used the transformation $y=uv-P/Q$ to transform equation (2) into

$$Q^2vu' = W - Q^2(v' - Pv)u - Q^3v^2u^2, \tag{3}$$

where $W=RQ^2+P'Q-PQ'$. If $W=0$, then v is chosen so that $v'-Pv=0$, and the solution to (3) is

$$u = 1 \Big/ \int Q\left(\exp\int Pdx\right)dx.$$

If $W>0$, then the variables in (3) can be separated provided that

$$\frac{-QW' + (3Q' + 2PQ)W}{2(-Q)^{1/2}W^{3/2}} = k,$$

where k is a constant. With v defined by $-Q^3v^2=W$, equation (3) then is written as

$$u' = (-W/Q)^{1/2}(1 - ku + u^2),$$

an equation in which the variables can be separated.

As a variation of the same idea, the following two results were presented in theorems by Rao and Ukidave [4]. First, if constants k, c and a function $v(x)$ exist such that $R=-cQv^2$ and $v'+Pv=kR$, then the transformation $y=vu$ reduces equation (2) to $u'=(-cQR)^{1/2}(1-ku+u^2/c)$. Second, if constants a, b and a function $v(x)$ exist such that $R+(P/Q)'=-aQv^2$ and $v'-Pv=b(-aQv^2)$, then the transformation $y=vu-P/Q$ reduces equation (2) to $u'=-Qv(a-abu+u^2)$.

Allen and Stein [5] found that if: R and Q are differentiable, $RQ<0$, and

$$\frac{-P + Q'/2Q - R'/2R}{(-RQ)^{1/2}} = c$$

where c is a constant, then equation (2) can be transformed by the substitution $y=(-R/Q)^{1/2}u$ into $u'=(-RQ)^{1/2}(1+cu+u^2)$, in which the variables can be separated.

A general plan of attack for finding conditions on P, Q, and R which will allow equation (2) to be transformed into an equation in which the variables can be separated has been presented by Wong [6]. He transformed the equation (2) by the substitution $y=uv-w$ and imposed the conditions that the nonhomogeneous term and the coefficient of u each be a constant multiplied by the coefficient of u^2. The result was the system

$$w' + Pw - Qw^2 + R = -aQv^2 \quad \text{and} \quad v' + Pv - 2wQv = bQv^2,$$

where a, b are constants. By picking particular solutions to this system one can find conditions which would thereby be imposed on P, Q, and R. If P, Q, and R satisfied the conditions, then equation (2) could be transformed into an equation in which the variables could be separated.

If the coefficients in (2) are polynomials, then particular polynomial solutions can sometimes be found. Murphy [2], Rainville [7], and others have presented such methods, but Campbell and Golomb [8] have developed a method by which polynomial solutions to the more general equation

$$Ay' = B_0 + B_1y + B_2y^2, \tag{4}$$

can be found. The coefficients A, B_0, B_1, and B_2 are polynomials in x which have respective degrees a, b_0, b_1, and b_2. If equation (4) has a polynomial solution, $y=\sum_{i=0}^{m} c_ix^i$, then the degree m of the polynomial must be in one of the following classes:

$$\text{(5)}\quad \begin{array}{lll} \text{(i)} & (b_0 - b_2)/2 & \\ \text{(ii)} & 1 + b_0 - a, \quad a - b_2 - 1 & \\ \text{(iii)} & b_0 - b_1, \quad b_1 - b_2 & \\ \text{(iv)} & 1 + b_0 - a = b_0 - b_1, \quad a - b_2 - 1 = b_1 - b_2, & \beta_1/\alpha. \end{array}$$

For class (iv), β_1 and α are the leading coefficients of B_1 and A, respectively; and all of the following conditions must hold:

(iv, a) $a - 1 = b_1$,

(iv, b) β_1/α is a positive integer,

(iv, c) $b_0 < b_1 + \beta_1/\alpha$, and

(iv, d) $b_2 < b_1 - \beta_1/\alpha$.

The above classes restrict the polynomials that one needs to try in order to find a solution. Of course, there may not be any polynomial solutions to equation (4). In addition, it should be noted that this method says nothing about possible polynomial solutions if the coefficients in equation (4) are not polynomials.

If particular solutions are found by any method, the general solution can always be found. There are three ways of doing this depending on whether one, two, or three particular solutions are known. These are listed as numbers 2, 3, and 4 in the table in Section 6.

3. Reduced form of the Riccati equation. The Riccati equation (2) may be written in the "reduced form,"

$$u' - u^2 = I, \tag{6}$$

by means of the transformation $y = -u/Q + (-P + Q'/Q)/2Q$. When the transformation $u = -w'/w$ is applied to equation (6), the result is the second order equation

$$w'' + Iw = 0. \tag{7}$$

The equation in (6) can also be transformed into the "canonical form,"

$$v' - Iv^2 = 1, \tag{8}$$

by means of the transformation $u = -1/v$. The I in (6), (7), and (8) represents a function of x in each case, and they are the same function in all three equation if the transformation indicated is used. Also, the equations in (7) and (8) are related by the transformation $v = w/w'$.

Nelson [**9**] proved that (6) can be transformed by $u = I^{1/2}z$ (Rao's transformation [**3**] applied to (6)) into an equation in which the variables are separable if and only if I is of the form

$$I = (b - ax)^{-2}. \tag{9}$$

He further proved that (8) can be transformed by $v = I^{-1/2}z$ (Rao's transformation [**3**]) into an equation in which the variables are separable if and only if I satisfies the same condition (9). He also proved that equation (7) can be transformed by a change in the independent variable into an equation with constant coefficients if and only if I satisfies condition (9). The substitution to be used in this case is $x = (b - e^{-at})/a$. This last result demonstrates the close relationship between second order linear homogeneous equations and the Riccati equations.

4. Transformations to second order equations. Sugai [10] presented two transformations which change the Riccati equation into a second order linear homogeneous equation (we shall present two new ones in Section 5). The transformation $y=u'/Qu$, which we shall call the *conventional transformation*, changes the Riccati equation (2) into the second order equation $u''+(P-Q'/Q)u'-QRu=0$.

Another transformation, $y=Ru/u'$, which we shall call *Sugai's transformation*, when applied to equation (2) results in the second order equation $u''-(P-R'/R)u'-QRu=0$. Sugai pointed out that both of these transformations can be developed by assuming a transformation of the form $y=uf/g$. By applying this transformation to (2) the result is

$$u'fg + uf'g - ufg' + Pufg + Qu^2f^2 - Rg^2 = 0. \tag{10}$$

$$\begin{matrix} 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 \end{matrix} \qquad \text{(Term numbers)}$$

One of u, f, and g is to be the new dependent variable and the other two are to be determined. By letting the sum of the third term and the fifth term be equal to zero, conditions are placed on u, f, and g. The result is the conventional transformation. Similarly, other conditions are placed on u, f, and g by setting the sum of term numbers two and six equal to zero. The result in this case is Sugai's transformation.

5. New methods. This section is based on the implications of setting the sum of certain terms in (10) equal to zero. We will let the notation $\{a, b\}$ represent setting the sum of term number a and term number b equal to zero. Thus, $\{3, 5\}$ indicates $-ufg'+Qu^2f^2=0$, from which we find that $uf/g=g'/Qg$. Since $y=uf/g$, we have obtained the conventional transformation (except that g is used instead of u). Similarly, $\{2, 6\}$ leads to Sugai's transformation. We will also consider setting the sum of more than two terms equal to zero, and will use a similar notation, $\{a, b, c\}$, $\{a, b, c, d\}$, etc., to indicate that the sum of terms a, b, and c is set equal to zero for $\{a, b, c\}$, etc.

Not all choices of $\{a, b\}$ lead to fruitful results; however, some of these choices lead in a natural way to some fairly simple tests for finding if a given Riccati equation can be solved by a particular method. We will go into some detail on two examples and treat some others in a less detailed way.

First, let us consider $\{4, 5\}$, which is the equation $Pufg+Qu^2g^2=0$. From this equation we find that

$$uf/g = -P/Q.$$

When we substitute $y=-P/Q$ into (2), we get $-P'/Q+PQ'/Q^2-P^2/Q+QP^2/Q^2=R$, which simplifies to

$$R = -(P/Q)'.$$

We have thus been led in a very natural way to the fact that any Riccati equation (2) which has coefficients that satisfy the condition $R=-(P/Q)'$ has

$y=-P/Q$ as a particular solution. The general solution can then be found (see number 2 in Section 6).

Next, consider $\{1, 4, 6\}$ which is the equation $u'fg+Pufg-Rg^2=0$. By solving for f and substituting into $y=uf/g$ we obtain the transformation

$$y = Ru/(u' + Pu). \tag{11}$$

This transformation, like the conventional transformation and Sugai's transformation, changes the Riccati equation into a second order linear homogeneous equation. The equation is $u''+(P-R'/R)u'+[P'-P(R'/R)-QR]u=0$, which may be written in the more useful form,

$$u'' + (P - R'/R)u' + R[(P/R)' - Q]u = 0. \tag{12}$$

Notice that if $Q=(P/R)'$, then equation (12) is a *first* order linear equation in u' and, therefore, can be solved. Thus we have found that any equation of the form

$$y' + Py + (P/R)'y^2 = R, \tag{13}$$

can be solved by means of the transformation (11).

There are several equations $\{a, b\}$ (or $\{a, b, c\}$) which lead to the solution of equation (13). We have shown that equation $\{1, 4, 6\}$, when solved for f, is one of them. Similarly, if the equation $\{2, 4, 6\}$ is solved for u, we obtain the transformation (11), except for a change in notation. This is natural since u and f occur symmetrically in terms 1 and 2 and identically in the other terms. The transformation (11) can also be obtained from equations $\{1, 2, 4, 6\}$, $\{1, 3, 4, 6\}$, and $\{2, 3, 4, 6\}$; however, only by means of a substitution. Equation $\{1, 2, 4, 6\}$, for example, results in the transformation $y=Ruf/[(uf)'+Puf]$. With w in place of uf we have the transformation (11). Note that each of these equations contains terms 4 and 6 in (10). If we solve $\{4, 6\}$ for u (or f or g) we get $y=R/P$. This turns out to be a particular solution of equation (13).

Another transformation which changes the Riccati equation into a second order equation is $y=(g'-Pg)/Qg$, the result of solving equation $\{3, 4, 5\}$ as a linear algebraic equation in u (or f). The second order equation is

$$g'' - (P + Q'/Q)g' - Q[(P/Q)' + R]g = 0,$$

which is first order linear in g' if $R=-(P/Q)'$. The equation $\{4, 5\}$ leads to the fact that $y=-P/Q$ is a particular solution to equation (2) when $R=-(P/Q)'$.

From $\{5, 6\}$ we find that $(R/Q)^{1/2}$ is a particular solution when $Q=R\exp(2\int Pdx)$ (or equivalently $P=(R/2Q)(Q/R)')$. In (10) we see that $\{5, 6\}$ implies $\{1, 2, 3, 4\}$; therefore, we might expect to get the same result using either one, and, in a sense, we do. The result of $\{1, 2, 3, 4\}$ is $y=\exp(-\int Pdx)$, which is also a particular solution when $Q=R\exp(2\int Pdx)$, since $(R/Q)^{1/2}$ and $\exp(-\int Pdx)$ are the same. However, such relations do not always occur in such an obvious way. For example, equation $\{1, 4, 6\}$ results in $y=Ru/(u'+Pu)$, while equation $\{2, 3, 5\}$ results in $y=u/\int Qudx$. The relation between these is not immediately apparent.

Section 6. Methods of solution of (2), $y' + Py + Qy^2 = R$.

	Conditions	Transformation	Resulting Equation	Remarks	Ref.
1	General	$y=u'/Qu$ $y=Ru/u'$ $y=Ru/(u'+Pu)$ $y=(u'-Pu)/Qu$	$u''+(P-Q'/Q)u'-QRu=0$ $u''-(P+R'/R)u'-QRu=0$ $u''+(P-R'/R)u'+R[(P/R)'-Q]u=0$ $u''-(P+Q'/Q)u'-Q[(P/Q)'+R]u=0$		[2] [10] {1, 4, 6} {3, 4, 5}
2	y_1 a solution	$y=y_1+1/z$	$z'-(P+2Qy_1)z=Q$	First order linear	[7]
3	y_1, y_2 solutions	$y=\dfrac{y_2u-y_1}{u-1}$	$u'=Q(y_2-y_1)u$	Variables separable	[2]
4	y_1, y_2, y_3 solutions	Cross ratio	$\dfrac{(y-y_1)(y_2-y_3)}{(y-y_2)(y_3-y_1)}=c$ (constant)	The resulting equation is the general solution.	[2]
5	P, Q, R constants			(2) is variables separable	[2]
6	$Q=0$			(2) is first order linear	[2]
7	$R=0$ (Bernoulli)	$y=1/z$	$z'-Pz=-Q$	First order linear	[2]
8	$a^2R=abP+b^2Q$ for some constants a, $b(a\neq 0)$			$y=b/a$ is a solution of (2). See 2.	
9	$\dfrac{-QW'+(3Q'+2PQ)W}{2(-Q)^{1/2}W^{3/2}}=k$ where $W=RQ^2+P'Q-PQ'\geqq 0$	$y=uv-P/Q$ v is defined by $v'-Pv=0$ if $W=0$, $v^2=-W/Q^3$ if $W>0$.	$u'=(-W/Q)^{1/2}(1-ku+u^2)$	Variables separable	[3]
10	k, c and $v(x)$ exist such that $R=-cQv^2$ and $v'+Pv=kR$	$y=vu$	$u'=(-cQR)^{1/2}(1-ku+u^2/c)$	Variables separable	[4]
11	a, b and $v(x)$ exist such that $R+(P/Q)'=-aQv^2$ and $v'-Pv=b(-aQv^2)$	$y=uv-P/Q$	$u'=-Qv(a-abu+u^2)$	Variables separable	[4]
12	$\dfrac{-P+Q'/2Q-R'/2R}{(-RQ)^{1/2}}=k$	$y=(-R/Q)^{1/2}u$	$u'=(-RQ)^{1/2}(1+ku+u^2)$	Variables separable	[5]

	Conditions	Transformation	Resulting Equation	Remarks	Ref.
13	$P=0$, $Q=-1$ and $R=(b-ax)^{-2}$	$y=R^{1/2}z$	$z'=(1-az+z^2)/(b-ax)$	Variables separable	[9]
14	$P=0$, $R=1$, and $Q=-(b-ax)^{-2}$	$y=(-Q)^{1/2}z$	$z'=(1+az+z^2)/(b-ax)$	Variables separable	[9]
15	$R=-(P/Q)'$ Special case of 11 and 13	$y=\dfrac{g'-Pg}{Qg}$	$(g')'-(P+Q'/Q)g'=0$	First order in g' also $y=-P/Q$ is a solution of (2). See 2.	{3, 4, 5} {4, 5}
16	$Q=(P/R)'$	$y=Ru/(u'+Pu)$	$(u')'+(P-R'/R)u'=0$	First order in u' also $y=R/P$ is a solution of (2). See 2.	{1, 4, 6} {4, 6}
17	$Q=R\exp(2\int P dx)$ Special case of 14			$y=(R/Q)^{1/2}$ is a solution of (2). See 2.	{5, 6}
18	Equation (1) if $m(2k+1)+4k=0$ for some $k \in \{0, 1, 2 \cdots\}$			Integrable in finite terms. See reference.	[2]
19	$xy'-ay+by^2=cx^n$ (a) $n=2a$ (b) $(n-2a)/2n=k>0$ (c) $(n+2a)/2n=k>0$	$y=x^a u$	$u'=x^{a-1}(c-bu^2)$	Variables separable See reference. See reference.	[2] [2] [2]
20	$Ay'=B_0+B_1y+B_2y^2$ Polynomial coefficients			Test solutions $y=\Sigma_{i=0}^{m} c_i x^i$ m in (5). Coefficients to be determined successively.	[8]
21	$y'=B_0+B_1y+y^2$ $\Delta=B_1^2-4B_0-2B_1'$ (a) Δ is even degree			$y=-1/2(B_1 \pm [\Delta^{1/2}])$ are the only possible polynomial solutions. $[D]$ is the polynomial part of D. If successful, use 2.	[7]
	(b) Δ is odd degree			No polynomial solutions.	[7]

Note. If the above methods do not apply, one may try various transformations of the dependent or independent variables to obtain other Riccati equations. The new equations may then be tested for the above conditions.

Prepared in an NSF Research Participation for High School Teachers Program, Colorado State University, by Mr. Haaheim under the direction of Professor Stein.

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ON AN ELEMENTARY INEQUALITY

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One of the simplest and most elegant accounts of the log and exponential functions is that given in the reference. There Landau derives all the properties of these functions using only a few facts about monotone sequences. In this note we take up the elementary and very useful inequality $1-x^{-1} \leqq \log x \leqq x-1$ as treated by Landau. We will show that equality occurs iff $x=1$. This, of course, is well-known, and may be proved using Rolle's Theorem. In Landau's book, Rolle's Theorem comes much later and this prompted the present proof, which is extremely simple and is entirely in the spirit of Landau's treatment.

For $x>0$ and $k=2^n$, let $a(n, x)=k(x^{1/k}-1)$. Then Landau shows that if $x>1$, $a(n, x)>a(n+1, x)>0$; if $x=1$, $a(n, x)=0$; and if $0<x<1$, $a(n, x)=-a(n, x^{-1})x^{1/k}$. Thus, as n tends to infinity, $a(n, x)$ tends to a limit denoted by $\log x$. What Landau failed to observe is that $a(n, x)>a(n+1, x)$ even if $0<x<1$. This will give us what we want. For $y \geqq 0$, $y^k-1 \geqq k(y-1)$, so that $x-1 \geqq a(n, x)$ for all $x>0$. If $x \neq 1$, $x-1 \geqq a(n, x)>a(n+1, x) \geqq \log x$. If $x=1$, we clearly have equality. The left hand inequality is equivalent to the right hand inequality.

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THE EQUATION OF A SPHERE

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A neat derivation for the equation of a circle passing through three given points was recently given by Bond (Math. Gazette, May 1966, Classroom Note 137). Here we extend the method to find the equation of a sphere passing through the four given points $P_r(x_r, y_r, z_r)$, $r=1, 2, 3, 4$.

It follows immediately that an equation of a two parameter family of spheres passing through the two points P_1 and P_2 is given by

$$L + \lambda M + \mu N = 0, \tag{1}$$

where

$$L(x, y, z) \equiv (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2),$$

$$M(x, y, z) \equiv \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}, \qquad N(x, y, z) \equiv \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

Here, $L=0$, $M=0$, $N=0$, are the equations, respectively, for a sphere with $\overline{P_1P_2}$ as a diameter and two planes passing through the sets of three points (P_1, P_2, P_3) and (P_1, P_2, P_4). λ and μ are now determined such that (1) is satisfied also by points P_3 and P_4. Whence, $\lambda = -L(P_4)/M(P_4)$, $\mu = -L(P_3)/N(P_3)$. (There's only one simple determinant to evaluate since $M(P_4)+N(P_3)=0$.)

We could, of course, have written down the equation of the sphere (or circle) immediately in terms of the determinantal equation

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0\cdot$$

But then there would be considerable computation involved in expanding out the determinant.

Bond's method can also be extended to determine the equation of a conic passing through the four given points $P_r(x_r, y_r)$, $r=1, 2, 3, 4$ and whose axes are parallel to the coordinate axes.

We start with the equation

$$\lambda(x - x_1)(x - x_2) + \mu(y - y_1)(y - y_2) + \nu R = 0 \tag{2}$$

where

$$R \equiv \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}\cdot$$

Here, $\lambda(x-x_1)(x-x_2)+\mu(y-y_1)(y-y_2)=0$ is the equation of a family of conics with axes parallel to the coordinate axes and for which $\overline{P_1P_2}$ is a diameter. The ratios $\lambda:\mu:\nu$ are now determined by requiring (2) to be also satisfied by points P_3 and P_4.

The conic could turn out to be a parabola and in that case its axis will only be parallel to one of the coordinate axes. It also may turn out that the last two equations for determining $\lambda:\mu:\nu$ are linearly dependent. In this case, there will be a family of conics with the desired property. This case will arise if we started out with four points which were given as the intersection of two conics of the appropriate type.

An alternative way of obtaining the desired conic is to start from the equation given by Loney (The Elements of Coordinate Geometry, Part I, Macmillan, London, 1954, pp. 378–379):

$$L_1(P_1, P_2)L_2(P_3, P_4) = \lambda L_3(P_1, P_4)L_4(P_2, P_3) \tag{3}$$

where $L_i(P_r, P_s)=0$ is an equation of the straight line containing points P_r and P_s. λ is now determined such that the xy term cancels out in (3).

A SIMPLE IRRATIONALITY PROOF FOR *n*th ROOTS OF POSITIVE INTEGERS

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In a paper to appear in the American Mathematical Monthly, M. V. Subbarao gave a proof of the irrationality of $\sqrt{N}$ for positive integers N not the square of another integer which depended essentially only on the well-ordering property of the natural numbers. He raised the question of whether a similar approach can be used to establish the irrationality of the nth root of a non-nth power integer for $n>2$. In 1964, E. A. Maier and Ivan Niven [1] gave a proof of this result which depended directly on the division algorithm and hence indirectly on the well-ordering property of the integers.

In this paper we give a new proof of the irrationality of such an nth root. The proof differs from the one given in [1] in that it is somewhat simpler and depends only on the well-ordering property and not on the division algorithm.

THEOREM. *If N is a positive integer and if $\sqrt[n]{N}$ is a rational number for $n\geqq 2$ then N is the nth power of an integer.*

Proof. Suppose $\sqrt[n]{N}$ is a rational number. Then let b be the smallest positive integer for which there exists an integer a such that

$$\sqrt[n]{N} = a/b.$$

Let J denote the set of positive integers and let I denote the set of integers. Set

$$S = \{k \in J : a^k/b \in I\}$$

where a and b are as above.

We now prove that $n-1\in S$. For if not then $a^{n-1}/b\notin I$. Hence there exists a unique $A\in I$ such that

$$A < a^{n-1}/b < A + 1.$$

Since $a^n = Nb^n$, we have

$$a^n - Aab = Nb^n - Aab$$

from which we obtain

$$a/b = (Nb^{n-1} - Aa)/(a^{n-1} - Ab).$$

But $0<a^{n-1}-Ab<b$ and this contradicts our choice for b. In particular the theorem has been proved for $n=2$.

Hence S is not empty and therefore it must have a least element, say m. Suppose $m>1$. Then $a^m/b\in I$ and $a^{m-1}/b\notin I$. Therefore there exists $K\in I$ such that $a^m=bK$ and there exists a unique $B\in I$ such that

$$B < a^{m-1}/b < B + 1.$$

Hence $a^m-Bab=bK-Bab$, from which we obtain $a/b=(K-Ba)/(a^{m-1}-Bb)$; but $0 <a^{m-1}-Bb<b$ which again contradicts our choice for b. Hence $m=1$ and the proof is complete.

Reference

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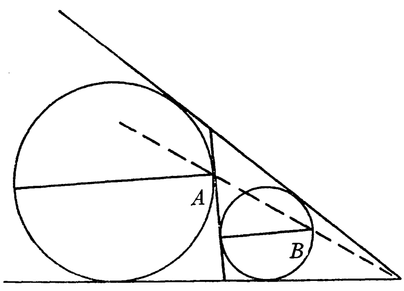
ANSWERS

A463. The intersection of the common external tangents of two circles is the external center of similitude. A and B are homologous points. Homologous points are collinear with the center of similitude.



A464. The problem makes sense for $-4\leqq x\leqq 4$. Since 0 and 4 are obviously solutions, we work with the intervals $A=(0, 4)$ and $B=(-4, 0)$. On A, the two sides of the equation differ by a constant c, since they have the same derivative. Setting $x=2$, we find that $c=0$, so that every x in A is also a solution.

To show that no x in B is a solution we need only note that $\sin^{-1}\{(x^2-8)/8\}$ is an even function while $2\sin^{-1}(x/4)$ is both odd and increasing, so that the fact that the graphs of the two sides coincide to the right of 0 shows that they can't even intersect to the left of 0. Thus the solution set is the interval [0, 4].

A465. If k is a nonnegative integer, then

$$\frac{(k+1)(k+2)\cdots(k+n)}{n!} = \frac{(k+n)!}{k!n!} = \binom{k+n}{n}$$

which is an integer. See Uspensky, *Elementary Theory of Numbers*, Page 100.

A466. Let A be $m\times n$ and B be $n\times m$ with $m\geqq n$. The rank of AB is m. But since the rank of a product of two matrices cannot exceed the rank of either of the two matrices which is at most n, we have $m\leqq n$. Thus $m=n$. The result is also valid if AB and BA are both nonsingular scalar matrices, e.g., if ABC, CAB and BCA are all nonsingular scalar matrices, then they are all square matrices.

A467. If a = person's age at the time of the SQUARED claim, b = year the person was born, then in order to make the SQUARED claim it must be true that $a^2=b+a$ at some time during the person's lifetime. Clearly a person can make the SQUARED claim at most once in his lifetime. From the equation $b=a^2-a$, construct the table:

a	1	2	3 $\cdots$	43	44	45
b	0	2	6	1806	1892	1980

Thus it appears that only a person born in 1892 could be alive today and make the SQUARED claim. He would have been 44 years old in the year $44^2=1936$.

(Quickies on page 277)

COIN STRINGS

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Most puzzlists have encountered the following coin problem: one is given three pennies (P) and two dimes (D), arranged $PDPDP$. One may take a penny and a dime which are adjacent and move them somewhere along the same line. The desired position is $PPPDD$. This can be achieved in four moves:

	$PDPDP$
1.	$PDP\quad PD$
2.	$P\quad PDPD$
3.	$PDPP\quad D$
4.	$PPPDD$

Consider the case of an *n-string*, a linear sequence of n dimes and $n+1$ pennies. Given *standard position* (pennies and dimes alternating), is it possible to achieve *bias position* (pennies at one end)? If so, how many moves will it take?

Start with the string in bias position with all the pennies at the right end (clearly, going from bias to standard position is equivalent to the reverse situation). There are two types of moves used: (1) Consider some dime which has a penny to its right. Move this pair to the right of the rightmost penny. This forms the triplet PDP at the right end of the string. (2) Move the PD part of this triplet to the hole created by the exit of the DP pair. For example:

	$DDD \cdots DDDDPPP \cdots PPPP$
move type (1)	$DDD \cdots DDD \quad PP \cdots PPPPDP$
move type (2)	$DDD \cdots DDDPDPP \cdots PPP \quad P$
move type (1)	$DDD \cdots DDDP \quad P \cdots PPP \quad PDP$
move type (2)	$DDD \cdots DDDPPDP \cdots PPP \quad\quad P$

These two moves, in conjunction, advance the dime one place to the right in the string—it "jumps" over one penny. We can perform $n-1$ such pairs of moves on the rightmost dime, consuming $2(n-1)$ moves, and producing the position: $DD \cdots DDP \cdots PPDP\ P$. Working on the second rightmost dime, we can go through the process $n-2$ times, spending $2(n-2)$ moves, and giving the position: $DD \cdots DDP \cdots PPDPDP\ P$. Continuing in this manner (and after $2+4+ \cdots +2(n-1)=n^2-n$ moves), $DPDP \cdots DPDP\ P$ will be obtained. By switching each of the n DP pairs to the right of the lonely penny, standard position is achieved. The total expenditure was n^2 moves, which agrees with the 2-string of the first paragraph.

Proving that any solution requires at least n^2 moves is a bit more difficult. A rigorous discussion would be too lengthy to present here, but the general attack can be given as follows: Suppose we desire to go from standard position to bias position with all the dimes at the right. (1) The argument hinges on the fact that each dime must eventually stop in final position. (2) It is less move-consuming to have the dime arrive in final position as a PD pair. If it arrives as a DP pair we have the problem of getting that penny to the left of the dime (since the pennies are to the left of the dimes in bias position), which is an expensive problem (move-wise). (3) Since every dime optimally arrives as a PD pair, the first dime to stop in final position is the rightmost, the next dime falls immediately to its left, and so forth. To make room for successive dimes, the penny in each PD pair must move. The conclusion follows, of course, that nothing less than n^2 moves suffices to make the transformation.

Whenever some move sequence transforms one string to another, the body of coins is displaced to the right or the left. In going from standard position to bias position in the 2-string, the whole body of coins slid six places to the right. More generally we can consider two string positions which are *move-equivalent* (one position can be obtained from the other through some set of moves), and ask what the displacement of this transformation is. It happens that the displacement separating two string positions is always the same, regardless of the particular moves used for the transformation. The argument here is fairly simple: (1) If one can go from one string position to another with two different displacements, then one can go from one string position *to itself* with some positive displacement (positive displacement being movement to the right; negative, to the left). (2) In any move, a penny and a dime move together an equal distance, so that after a sequence of moves the sum of the distances moved by the pennies must equal that of the dimes. However, since there is one more penny than there are dimes, any positive displacement of the string must result in a greater total movement by the pennies. Thus, displacement is unique.

Suppose we are given some string move-equivalent to standard position. Number the coins in this position from 1 to $2n+1$, and suppose the sum of the numbers assigned to the pennies is k. Then the displacement of this string from standard position is $2((n+1)^2-k)$ (the proof is direct and uses the ideas developed in the last paragraph). By fixing the displacement of standard position at 0, any other string has associated with it the unique number given by the formula above.

The assumption of the last two paragraphs is that the two strings are move-equivalent: there can be no displacement between two strings if you cannot obtain one from the other. The problem of move-equivalence is broken into several parts.

1. Suppose a given string has displacement D and is move-equivalent to standard position. It is clear that its mirror-image has a displacement of $-D$ and is also move-equivalent to standard position (the proof is not difficult). This implies that to examine those string positions which are move-equivalent to standard position we have only to consider strings with negative displacement.

2. Suppose some string with negative displacement does not have a penny as its rightmost coin. There is a string of dimes at the right end. Switch the PD pair with some other DP pair in the string:

$$PPP \cdots DP \cdots PPDDDD$$

$$PPP \cdots PD \cdots PDPDDD$$

This move is clearly legitimate and reduces by one the number of dimes at the right end. Suppose there is no DP pair in the string. Absence of such a DP pair would imply that the string had positive displacement, in contradiction to the original assumption. Hence, any string with negative displacement is move-equivalent to one with a penny as its rightmost coin.

3. Starting with the string in this position (negative displacement, penny as the rightmost coin), move the dimes to the right end of the string, using the pairs of moves given to transform bias position to standard position. This will also position the rightmost penny somewhere to the right, and the string position will be $PPPPP \cdots PDD \cdots DDD\ P$. Now begin to move the dimes leftward with the inverse type of move:

$$PPPPP \cdots PPDD \cdots DDD \qquad P$$

$$PPPPP \cdots P \quad D \cdots DDD \quad PDP$$

$$PPPPP \cdots PDPD \cdots DDD \quad P$$

Continue in this fashion until $PDPDPD \cdots PDPD\ P$ is reached. The PD pairs can be slid rightward and standard position is obtained. Any string of this type, then, is move-equivalent to standard position; thus, all string positions are move-equivalent.

This completely covers the case of move-equivalence when the pennies (and the dimes) are considered indistinguishable. If you place the coins in standard position, and assign numbers to them from 1 to $2n+1$, there now arise $(2n+1)!$ distinct string positions. Any move plainly results in an even permutation of these numbers, since at the same time that a penny jumps over t numbers, its accompanying dime jumps over t numbers. Straightforward analysis reveals that with the coins in standard position every even permutation is obtainable. The results of the last section imply that move-equivalence partitions the set of all numbered strings into two groups, with any member of one group an odd permutation of every element of the other.

A PROPERTY OF THE ZEROS OF A POLYNOMIAL

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Introduction. When using the Newton-Raphson method to find the roots of an equation, it is of interest to find "good" starting points for the iteration. Clearly, the "best" starting points would be those which would give a root at the first try. The following theorem gives an interesting relationship between the zeros of a polynomial and the abscissas of the "best" starting points, i.e., the abscissas of those points on the polynomial where a tangent passes through one of the zeros of the polynomial.

THEOREM. *Let*

$$y = f(x) = \sum_{i=0}^{n} a_i x^i = a_n \prod_{i=1}^{n} (x - r_i). \tag{1}$$

Let $t_{k1}, t_{k2}, \cdots, t_{k,n-2}$ *be the abscissas of the points of tangency of the tangents to* $y=f(x)$ *through* $(r_k, 0)$, $k=1, 2, \cdots, n$. *Then*

$$\sum_{i=1}^{n-2} t_{ki} = \frac{n-2}{n-1} \sum_{i=1}^{n}{}' r_i, \tag{2}$$

where $\sum'$ *indicates that* r_k *is excluded from the summation.*

Proof. Let $y=m(x-r_k)$ be the equation of an arbitrary straight line through the zero $(r_k, 0)$. The abscissas of the points of intersection of this line with the polynomial $y=f(x)$ are the solutions of the equation:

$$m(x - r_k) = a_n \prod_{i=1}^{n} (x - r_i). \tag{3}$$

Since we are looking for abscissas different from $x=r_k$, we can divide both sides by $(x-r_k)$ and obtain:

(4) $$m = a_n(x - r_1)(x - r_2) \cdots (x - r_{k-1})(x - r_{k+1}) \cdots (x - r_n)$$

or

(5) $$x^{n-1} - (r_1 + r_2 + \cdots + r_{k-1} + r_{k+1} + \cdots + r_n)x^{n-2} + \cdots + \left[(-1)^{n-1}r_1r_2 \cdots r_n - \frac{m}{a_n}\right] = 0.$$

For a general line this will give rise to $(n-1)$ solutions. If the line is to be a tangent to $y=f(x)$, the derivative of (5) must also be $=0$ (criterion for multiple root). We must have, therefore,

$$(n - 1)x^{n-2} - (n - 2)(r_1 + r_2 + \cdots + r_{k-1} + r_{k+1} + \cdots + r_n)x^{n-3} + \cdots = 0.$$

If we denote the $(n-2)$ roots of this equation by $t_{k1}, t_{k2}, \cdots, t_{k,n-2}$, then, by a well-known result from the theory of equations, we have:

$$\sum_{i=1}^{n-2} t_{ki} = \frac{n-2}{n-1}(r_1 + r_2 + \cdots + r_{k-1} + r_{k+1} + \cdots + r_n)$$

which is (2).

COROLLARY. *For the cubic* $y=c(x-x_1)(x-x_2)(x-x_3)$, *the tangent at the point with abscissa* $(x_1+x_2)/2$ *passes through* $(x_3, 0)$.

THE COSET OF SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS

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There is a natural relationship between two topics which usually are given thorough treatment in the undergraduate mathematics curriculum—the solution of a system of linear equations over a field and the fundamental homomorphism theorem for groups. The relationship between these two topics is so accessible that it seems unfortunate that so few (if any) textbook authors (and perhaps then instructors) give the undergraduate algebra student the opportunity to discover it.

In linear algebra the problem of solving the system

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad i = 1, 2, \cdots, m, \; a_{ij} \in F, \text{ a field,}$$

is considered, and the student discovers that this problem is equivalent to solving the matrix equation $Ax=b$, where $x=(x_1, x_2, \cdots, x_n)$ in $V_n(F)$,

$b=(b_1, b_2, \cdots, b_m)$ in $V_m(F)$, and A is the matrix of a linear transformation $T_A\colon V_n(F)\to V_m(F)$. He further discovers that $T_A(V_n(F))$ is a subspace of $V_m(F)$, and that the system has a solution if and only if $b\in T_A(V_n(F))$. Finally, he discovers that the set of solutions (if it exists) can be described as $\{u+v\}$ where u is one particular solution to the system $Ax=b$ and v is any solution to the homogeneous system $Ax=0$.

In algebraic structures the fundamental homomorphism theorem for groups is considered, and the student discovers that if $\phi:G\to G'$ is a homomorphism from a group G into a group G', then $\phi(G)$ is a subgroup of G' and (among other results) that two elements in G have the same image in $\phi(G)$ if and only if they are members of the same coset of G modulo K, where K is the kernel of the homomorphism ϕ.

Rarely, if ever, do textbooks currently being used in courses in algebraic structures indicate the connection between these two topics. That is, if one examines $T_A\colon V_n(F)\to V_m(F)$, it is immediately recognized that T_A can be considered as a group homomorphism (between the two additive groups $V_n(F)$ and $V_m(F)$) whose range is the subgroup $T_A(V_n(F))$ and whose kernel is $H=\{x\mid Ax=0\}$, the solution set of the homogeneous system. Finally, then, using the fundamental homomorphism theorem, it is seen that the solution of the system $Ax=b$ (if it exists) is a coset $u+H$, where u is any representative (i.e., any particular solution of the system $Ax=b$).

ITERATION AND CORRECTION FORMULAS FOR THE VARIANCE OF A SEQUENCE

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Even though a computer program is usually available for obtaining the mean and variance of a sequence, formulas for finding the new mean and variance if more data point(s) are to be inserted or old data point(s) are to be corrected would be useful. A very efficient computer program could also be written utilizing these formulas which would not necessitate storing the data since it would require only one data point per iteration.

Before deriving the iteration formulas we introduce the following notation:

$\mu_n=$ the mean of the sequence $x_1, x_2, \cdots, x_n$

$\sigma_n^2=$ the variance of the sequence $x_1, x_2, \cdots, x_n$

$\mu_{m,n}=$ the mean of the sequence $x_{n+1}, x_{n+2}, \cdots, x_{n+m}$

$\sigma_{m,n}^2=$ the variance of the sequence $x_{n+1}, x_{n+2}, \cdots, x_{n+m}$.

I. For introducing only one new value or correcting for a single value in the sequence.

From the definition of the mean we easily obtain the iteration formula:

$$\mu_{n+1}=\left(\frac{n}{n+1}\right)\mu_n+\left(\frac{1}{n+1}\right)x_{n+1} \tag{1}$$

where $\mu_0=0$ by definition.

To obtain the variance we use the form:

$$
\begin{aligned}
\sigma_{n+1}^2 &= \left(\frac{1}{n+1}\right)\sum_{i=1}^{n+1} x_i^2 - \mu_{n+1}^2 \\
&= \left(\frac{n}{n+1}\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \mu_n^2\right) + \left(\frac{n}{n+1}\right)\mu_n^2 + \left(\frac{1}{n+1}\right)x_{n+1}^2 - \mu_{n+1}^2
\end{aligned} \tag{2}
$$

and substituting for μ_{n+1} from (1)

$$
\begin{aligned}
\sigma_{n+1}^2 &= \left(\frac{n}{n+1}\right)\sigma_n^2 + \frac{n\mu_n^2 + x_{n+1}^2}{n+1} - \frac{(n\mu_n + x_{n+1})^2}{(n+1)^2} \\
&= \left(\frac{n}{n+1}\right)\sigma_n^2 + \frac{n(n+1)\mu_n^2 + x_{n+1}^2(n+1) - (n^2\mu_n^2 + 2nx_{n+1}\mu_n + x_{n+1}^2)}{(n+1)^2} \\
&= \left(\frac{n}{n+1}\right)\sigma_n^2 + \frac{n\mu_n^2 - 2n\mu_n x_{n+1} + nx_{n+1}^2}{(n+1)^2} \\
&= \left(\frac{n}{n+1}\right)\sigma_n^2 + \frac{n}{(n+1)^2}(\mu_n - x_{n+1})^2.
\end{aligned} \tag{3}
$$

We define $\sigma_0^2 = 0$ making (3) usable for $n = 0, 1, 2 \cdots$

(3) along with (1) are used to correct for a value in error, say x_{n+1} as follows. Rewriting (1) in reverse we get

$$
(\mu_n)_c = \left(\frac{n+1}{n}\right)\mu_{n+1} - \left(\frac{1}{n}\right)x_{n+1}. \tag{4}
$$

Now (4) gives us the correct mean for the n correct values in terms of the incorrect mean μ_{n+1} and the incorrect value x_{n+1}, allowing us to iterate backwards to the correct mean. Next to obtain the correct mean $(\mu_{n+1})_c$, suppose $(x_{n+1})_c$ is the correct value for x_{n+1}, then (1) is used to give

$$
(\mu_{n+1})_c = \left(\frac{n}{n+1}\right)(\mu_n)_c + \left(\frac{n}{n+1}\right)(x_{n+1})_c. \tag{5}
$$

Next (3) in reverse gives us

$$
(\sigma_n^2)_c = \left(\frac{n+1}{n}\right)\sigma_{n+1}^2 - \left(\frac{1}{n+1}\right)\left[x_{n+1} - (\mu_n)_c\right]^2 \tag{6}
$$

the variance for the n correct values in terms of the incorrect value x_{n+1} and variance σ_{n+1}^2, along with the correct mean $(\mu_n)_c$. Finally, (3) is used to obtain the correct variance $(\sigma_{n+1}^2)_c$

$$
(\sigma_{n+1}^2)_c = \left(\frac{n}{n+1}\right)(\sigma_n^2)_c + \frac{n}{(n+1)^2}\left[(x_{n+1})_c - (\mu_n)_c\right]^2 \tag{7}
$$

where $(x_{n+1})_c$ is the correct value to be inserted for x_{n+1} and $(\mu_n)_c$ is obtained as previously shown.

Possible uses for the above formulas:

(1) Obtaining the mean and variance when one more sample value must be inserted, without recomputing the variance for the whole sample.
(2) Determining the effect of additional values on the mean and variance.
(3) Correcting the mean and variance when an error has been detected, without recomputing for the new sample after replacing the one in error by its correct value.
(4) Determining the effect of an error in a value on the mean and variance.

II. For introducing m new values or correcting for m values in the sequence. Proceeding from the definition of the mean

$$
\begin{aligned}
\mu_{n+m} &= \frac{1}{n+m}\sum_{i=1}^{n+m} x_i \\
&= \left(\frac{n}{n+m}\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \frac{m}{n+m}\left(\frac{1}{m}\sum_{n+1}^{n+m} x_i\right) \qquad (8)\\
&= \frac{n\mu_n + m\mu_{m,n}}{n+m},
\end{aligned}
$$

which gives the overall mean in terms of the mean of the first n values and the mean of the next m values.

$$
\begin{aligned}
\sigma^2_{n+m} &= \frac{1}{n+m}\sum_{i=1}^{n+m} x_i^2 - \mu^2_{n+m} \\
&= \left(\frac{n}{n+m}\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \mu_n^2\right) + \left(\frac{m}{n+m}\right)\left(\frac{1}{m}\sum_{i=n+1}^{n+m} x_i^2 - \mu^2_{m,n}\right) \\
&\quad + \frac{n\mu_n^2 + m\mu^2_{m,n}}{n+m} - \left(\frac{n\mu_n + m\mu_{m,n}}{n+m}\right)^2 \qquad (9)\\
&= \frac{n}{n+m}\sigma_n^2 + \frac{m}{n+m}\sigma^2_{m,n} + \frac{nm\mu_n^2 - 2nm\mu_n\mu_{m,n} + nm\mu^2_{m,n}}{(n+m)^2} \\
&= \frac{n}{n+m}\sigma_n^2 + \frac{m}{n+m}\sigma^2_{m,n} + \frac{nm}{(n+m)^2}\left(\mu_n - \mu_{m,n}\right)^2.
\end{aligned}
$$

Note that for a single value in the new sequence $\sigma^2_{1,n}=0$, $\mu_{1,n}=x_{n+1}$ so that (9) reduces to (3). Similarly (8) reduces to (1).

Iteration formulas (8) and (9) can be used in the same way as (3) and (1) in which the iteration and correction can take place on a set of m values instead of only one.

III. If the unbiased form of the variance is to be used where

$$S_n^2 = \left(\frac{1}{n-1}\right)\sum_{i=1}^{n}\left(x_i - \mu_n\right)^2$$

$$S_{m,n}^2 = \left(\frac{1}{m-1}\right)\sum_{i=m+1}^{n+m}(x_i - \mu_{m,n})^2$$

$$S_{n+m}^2 = \left(\frac{1}{n+m-1}\right)\sum_{i=1}^{n+m}(x_i - \mu_{n+m})^2$$

then the iteration formulas for S_{n+m}^2 are obtained using the relationships

$$S_n^2 = \frac{n}{n-1}\sigma_n^2$$

$$S_{m,n}^2 = \frac{m}{m-1}\sigma_{m,n}^2$$

$$S_{n+m}^2 = \frac{n+m}{n+m-1}\sigma_{n+m}^2$$

and substituting into (9). The resulting iteration formula is:

$$\text{(10)} \quad S_{n+m}^2 = \frac{n-1}{n+m-1}S_n^2 + \frac{m-1}{n+m-1}S_{m,n}^2 + \frac{mn}{(n+m)(n+m-1)}(\mu_n - \mu_{m,n})^2.$$

ON PALINDROMES AND PALINDROMIC PRIMES

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1. Palindromes. Let N' be the integer obtained by writing the digits of the base ten integer N in reverse order, and let

$$N + N' = S_1, \qquad S_1 + S_1' = S_2, \cdots, S_k + S_k' = S_{k+1}.$$

It has been conjectured that for each N there is a k such that S_k is palindromic. This conjecture is discussed in [1] and [4]. In [4] it is noted, for example, that $S_k(196)$ is not palindromic for $k \leqq 4147$, but no regularity or periodicity in the patterns of digits is apparent.

Although it seems difficult to prove or disprove the above conjecture for base ten integers, it may be easier to recognize certain regularities or repetitive patterns for smaller bases. In fact, the following counterexample proves that the conjecture is false for base two integers:

Let N be the base two integer 1101000101. Then

$$S_4(N) = 110010001101,$$

$$S_8(N) = 11000100011101,$$

and it is not difficult to show that for any positive integer i,

$$S_{4i}(N) = \underbrace{110 \cdots 0}_{\binom{i+1}{\text{zeros}}}\underbrace{10001 \cdots 1}_{\binom{i+1}{\text{ones}}}01$$

and that $S_j(N)$ is not palindromic for $j=4i+1,\ 4i+2,\ 4i+3$.

Questions. Is the conjecture false for base three integers? Is the conjecture false for every base greater than one?

2. Palindromic primes. It is easy to prove that every palindrome composed of an even number of digits is divisible by 11. The following observations suggest that there may be infinitely many palindromic primes and, moreover, that there may be relatively more primes in the set of palindromes composed of an odd number of digits than there are primes in the set of positive integers composed of an odd number of digits.

Let n be a positive odd integer and define:

A_n = the number of positive palindromes composed of n digits ($A_n = 9\cdot 10^{(n-1)/2}$)

α_n = the number of palindromic primes composed of n digits

B_n = the number of positive integers composed of n digits ($B_n = 9\cdot 10^{n-1}$)

β_n = the number of primes composed of n digits

$$P(n) = \frac{\alpha_n}{A_n} \Big/ \frac{\beta_n}{B_n} = 10^{(n-1)/2}\left(\frac{\alpha_n}{\beta_n}\right).$$

An examination of [2] reveals that

$$P(1) = \frac{4}{4} = 1,$$

$$P(3) = 10\left(\frac{15}{143}\right) = 1.049,$$

$$P(5) = 10^2\left(\frac{93}{8363}\right) = 1.112,$$

$$P(7) = 10^3\left(\frac{668}{586081}\right) = 1.140$$

This suggests the possibility that, for odd n, $P(n)$ is an increasing function of n. We note also that for large values of n, the prime number theorem (see [3]) provides a method of approximating β_n.

Questions. Is $P(n) > 1$ for every odd value of $n > 1$? Is $P(n)$ an increasing function of odd values of n? How may $P(n)$ (or, equivalently, α_n) be approximated for large odd values of n?

Similar questions may be investigated for bases other than ten. It may be noted, for example, that the infinitude of primes of the form $2^n - 1$ would imply the infinitude of base two palindromic primes.

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PALINDROMES BY ADDITION IN BASE TWO

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In his article in this magazine entitled *Palindromes by Addition* [1], Charles W. Trigg explored a conjecture regarding the production of palindromes by a certain type of addition in base ten. Given a positive integer N, let the integer N' be the integer N with digits reversed. Form $N+N'=S_1$. If S_1 is palindromic, the conjecture has been verified in this case. If not, take $S_1+S_1'=S_2$, and so on. For certain integers, such as 196, this process has been carried out thousands of times without producing a palindromic pattern. Moreover, no form of periodicity was observed which would indicate that a palindromic pattern was impossible.

Such being the situation, the author decided to test the conjecture in base two. Already at 22, a pattern was found which showed that no palindromic number would ever occur in the sequence S_i. The successive values N, S_1, $S_2, \cdots, S_{20}$ are shown below.

1	0	1	1	0								
1	0	0	0	1	1							
1	0	1	0	1	0	0						
1	1	0	1	0	0	1						
1	0	1	1	0	1	0	0					
1	1	1	0	0	0	0	1					
1	0	1	1	0	1	0	0	0				
1	1	0	0	1	0	1	0	1				
1	0	1	1	1	0	1	0	0	0			
1	1	0	1	0	0	0	1	0	1			
1	0	1	1	1	0	1	0	0	0	0		
1	1	0	0	0	1	0	1	1	0	1		
1	0	1	1	1	1	0	1	0	0	0	0	
1	1	0	0	1	0	0	0	1	1	0	1	
1	0	1	1	1	1	0	1	0	0	0	0	0

1 1 0 0 0 0 1 0 1 1 1 0 1
1 0 1 1 1 1 1 0 1 0 0 0 0 0
1 1 0 0 0 1 0 0 0 1 1 1 0 1
1 0 1 1 1 1 1 0 1 0 0 0 0 0 0
1 1 0 0 0 0 0 1 0 1 1 1 1 0 1
1 0 1 1 1 1 1 1 0 1 0 0 0 0 0 0

Note, for example, S_{20}, S_{16}, and S_{12}. In abbreviated form they could be written (the subscript indicating the number of times the digit is to be taken in succession):

$$S_{12} = 1\ 0\ 1_4\ 0\ 1\ 0_4$$
$$S_{16} = 1\ 0\ 1_5\ 0\ 1\ 0_5$$
$$S_{20} = 1\ 0\ 1_6\ 0\ 1\ 0_6$$

That this pattern will continue to repeat itself indefinitely may be demonstrated as follows. We start with 101_r010_r expressing this and subsequent integers in a form convenient for carrying out the process of addition.

$$\begin{array}{llllllllll}
1 & 0 & 1_{r-2} & 1 & 1 & 0 & 1 & 0_{r-2} & 0 & 0 \\
0 & 0 & 0_{r-2} & 1 & 0 & 1 & 1 & 1_{r-2} & 0 & 1 \\
\hline
1 & 1 & 0_{r-2} & 1 & 0 & 0 & 0 & 1_{r-2} & 0 & 1 \\
1 & 0 & 1_{r-2} & 0 & 0 & 0 & 1 & 0_{r-2} & 1 & 1 \\
\hline
1 & 0 & 1_{r-1} & 1 & 0 & 1 & 0_{r-1} & 0 & 0 & \\
0 & 0 & 0_{r-1} & 1 & 0 & 1 & 1_{r-1} & 0 & 1 & \\
\hline
1 & 1 & 0_{r-1} & 0 & 1 & 0 & 1_{r-1} & 0 & 1 & \\
1 & 0 & 1_{r-1} & 0 & 1 & 0 & 0_{r-1} & 1 & 1 & \\
\hline
1 & 0 & 1_{r+1} & 0 & 1 & 0_{r+1} & & & &
\end{array}$$

This particular pattern was quite prevalent, being found in 24% of the cases for integers up to 650.

A second period pattern was noted for the first time at $N=77$. The period again is four. Two periods are shown below.

1 0 1 1 1 0 0 0 1 0 0 0 0
1 1 0 0 0 0 0 1 0 1 1 0 1
1 0 1 1 1 0 1 0 1 1 0 0 0 0
1 1 0 0 1 0 0 0 0 0 1 1 0 1
1 0 1 1 1 1 0 0 0 1 0 0 0 0 0
1 1 0 0 0 0 0 0 1 0 1 1 1 0 1
1 0 1 1 1 1 0 1 0 1 1 0 0 0 0 0
1 1 0 0 0 1 0 0 0 0 0 1 1 1 0 1
1 0 1 1 1 1 1 0 0 0 1 0 0 0 0 0 0

As in the previous case it is possible to prove that this pattern will continue to generate itself.

A third pattern ($N=775$, e.g.) was found, again of period four. Two periods are shown below.

1	0	1	1	1	0	0	1	0	0	0	1	0	0	0	0				
1	1	0	0	0	0	0	1	1	0	1	0	1	1	0	1				
1	0	1	1	1	0	1	1	1	0	0	1	1	0	0	0	0			
1	1	0	0	1	0	0	0	1	0	0	0	0	1	1	0	1			
1	0	1	1	1	1	0	0	1	0	0	0	1	0	0	0	0	0		
1	1	0	0	0	0	0	0	1	1	0	1	0	1	1	1	0	1		
1	0	1	1	1	1	0	1	1	1	0	0	1	1	0	0	0	0	0	
1	1	0	0	0	1	0	0	0	1	0	0	0	0	1	1	1	0	1	
1	0	1	1	1	1	1	0	0	1	0	0	0	1	0	0	0	0	0	0

Up to 650, the highest sequence member with a palindromic pattern was S_{11}. There were also some dozen cases in which no palindromic pattern was found and no periodic property observed.

Reference

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ON ANALYTIC FUNCTIONS SATISFYING THE MEAN VALUE THEOREM AND A CONJECTURE OF W. G. DOTSON

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In his note [1] W. G. Dotson, Jr., proved among others that a polynomial $P(z)$ with complex coefficients satisfies the complex analogue of the classical mean value theorem for real differentiable functions if and only if its degree does not exceed two. He also conjectured that the above theorem holds if we include all the entire transcendental functions. The purpose of this note is to indicate a few results concerning this problem and to show that a considerably stronger result holds.

We begin with the following:

DEFINITION. *Let $f(z)$ be analytic at the point $z=a$. We say that $f(z)$ satisfies the mean value theorem at a if there exists a neighborhood $N(a)$ of a such that*

$$f(z)-f(a)=(z-a)f'(\zeta)$$

for all $z\in N(a)$ and $\zeta=\zeta(z)$ is a point on the open line segment joining a and z.

We prove now: LEMMA 1. *If $f''(a)\neq 0$, then $f(z)$ does not satisfy the mean value theorem at a unless $f(z)$ is a polynomial of degree 2.*

Proof. By considering if necessary $g(z)=f(z+a)-f(a)$ we may assume that $a=0$ and that $f(0)=0$. Thus let

$$f(z) = zf'(z\eta(z)) \tag{1}$$

be satisfied for all z in a neighborhood of the origin with $0<\eta(z)<1$. Since $f''(0)\neq 0$ the function $f'(z)$ maps a sufficiently small neighborhood of the origin one to one conformally onto a neighborhood of $f'(0)$. Therefore for sufficiently small $|z|$ equation (1) yields

$$\eta(z) = \frac{1}{z}f'^{-1}\left(\frac{f(z)}{z}\right). \tag{2}$$

Equation (2) implies that $\eta(z)$ is a single-valued analytic function at $z=0$. Since $\eta(z)$ is an open mapping unless constant we therefore conclude that $\eta(z)\equiv c$.

Therefore

$$f(z) = zf'(cz). \tag{3}$$

If the Taylor series of $f(z)$ is

$$f(z) = a_1z + a_2z^2 + \cdots + a_nz^n + \cdots$$

then comparing coefficients in equation (3) we obtain

$$nc^{n-1}a_n = a_n, \qquad n = 1, 2, \cdots. \tag{4}$$

Since $a_2\neq 0$, we have $c=\frac{1}{2}$ and $a_n=0$ for $n\geqq 3$. Therefore $f(z)$ is then a polynomial of degree 2. This completes the proof.

Remark 1. It is clear from the proof that we may considerably relax the conditions on $\eta(z)$, e.g., we may only require that $\eta(z)$ be bounded and omit a fixed argument. Correspondingly, we may relax the definition of the mean value property to allow ζ to lie inside any circle passing through a and z.

Remark 2. The condition $f''(a)\neq 0$ is necessary to prove the theorem as the example

$$f(z) = z + a_nz^n, \qquad a_n \neq 0,\ n > 2$$

shows.

This polynomial satisfies (3) with $c=n^{-1/(n-1)}$ and therefore satisfies the mean value property at the origin. On the other hand the linear functions satisfy the mean value property at every point and their second derivative vanishes identically.

We continue with the investigation of the case $f''(0)=0$.

LEMMA 2. *If $f(z)$ (not identically zero) is normalized as in Lemma 1 and $f''(0)=0$ and if $f(z)$ satisfies the mean value theorem at the origin, then $f(z)$ is a polynomial of the form*

$$p(z) = a_1 z + a_n z^n \qquad (n \neq 2). \tag{5}$$

Proof. Since $f''(0)=0$, there exists a deleted neighborhood of the origin $|z|<\epsilon$ such that $f''(z)\neq 0$ for $0<|z|<\epsilon$. Consider again equation (1) which we may assume to be satisfied in $|z|<\epsilon$.

We consider the following possibilities:

(a) $\eta(z)\equiv 0$ in some neighborhood contained in $|z|<\epsilon$. Then obviously $f(z)$ is a multiple of z.

(b) $\eta(z)\equiv c$, $c\neq 0$, in some neighborhood contained in $|z|<\epsilon$.

Then equation (3) is an identity for all $|z|<\epsilon$. If $f(z)$ is not a linear function, then

$$f(z) = a_1 z + a_k z^k + a_{k+1} z^{k+1} + \cdots, \qquad a_k \neq 0,\ k > 2.$$

Equalities (4) yield $kc^{k-1}=1$ and $a_l=0$ for $l>k$. It remains to show that (a) or (b) exhaust all the possibilities. Indeed choose an arbitrary point z_0, $0<|z_0|<\epsilon$ for which $\eta(z_0)\neq 0$, and let $\zeta_0=z_0\eta(z_0)$. Then $0<|\zeta_0|<\epsilon$ and thus $f''(\zeta_0)\neq 0$. It follows as in Lemma 1 that in a small neighborhood of ζ_0 we have the relation (2) which implies that $\eta(z)$ satisfies the condition (b). This completes the proof of Lemma 2.

Since it is now a matter of a simple verification to check that the polynomial (5) does not satisfy the mean value property at any point different from the origin if its degree is greater than 2, we have:

THEOREM. (a) *The polynomials of degree not exceeding two satisfy the mean value theorem at all points of the complex plane.*

(b) *Any polynomial of degree greater than 2 can satisfy the mean value property in at most one point.*

(c) *If a polynomial satisfies the mean value property at a point $z=a$ then it is of the form*

$$f(z) = a_0 + a_1(z-a) + a_n(z-a)^n.$$

(d) *An analytic transcendental function in a domain $\mathfrak{D}$ does not satisfy the mean value property at any point of $\mathfrak{D}$.*

In spite of the somewhat discouraging results regarding the local mean value property it would be interesting to investigate the range of possible values of $\eta(z)$ such that equation (1) holds in a neighborhood of the origin. If, for example, we consider the class of all polynomials of degree n vanishing at the origin, equation (1) is a polynomial of degree at most $n-1$ in η for each fixed z and therefore, except in the trivial case, admits at most $n-1$ and at least one solution.

One can obtain some estimate of this range as follows:

Let $f(z)=z+a_2z^2+\cdots+a_nz^n$, $a_n\neq 0$. The expression $f(z)-zf'(z\eta(z))$ is linear and symmetric in the zeros $z_1, z_2, \cdots, z_n$ of $f(z)$ of total degree n unless $n\eta^{n-1}=1$.

Therefore by a result due to J. L. Walsh [2], there exists a function $\alpha(z)$ whose values lie in any circular domain containing all the zeros of $f(z)$ such that

$$(z - \alpha(z))^n - nz(z\eta - \alpha(z))^{n-1} = 0.$$

This obviously yields some results about the range of $\eta(z)$ which we plan to investigate separately.

The author acknowledges partial support by NSF Grant GP-5221.

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THE LANGUAGE OF FUNCTIONS—A SURVEY AND A PROPOSAL

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1. Introduction. From the point of view of Bourbaki [**3**], a *correspondance* (in English usually called a *relation*) is an ordered triple of sets $\Gamma = (G, A, B)$ where A and B are arbitrary sets and G is a subset of $A \times B$. The set A is called *l'ensemble de départ de* Γ, B is called *l'ensemble d'arrivée de* Γ, and G is called *le graphe de* Γ. While only these three sets appear in the definition, there are two others that play a role in the theory. These are

$$pr_1G = \{x \mid x \in A, (x, y) \in G \text{ for some } y \in B\}, \text{ called}$$

l'ensemble de définition de Γ, and

$$pr_2G = \{y \mid y \in B, (x, y) \in G \text{ for some } x \in A\}, \text{ called}$$

l'ensemble des valeurs de Γ,

In order for a relation $f = (F, A, B)$ to be a *function* it is necessary that

(1) *l'ensemble de départ = l'ensemble de définition,*

(2) to each $x \in A$ there corresponds at most one object in B.

The purpose of this paper is to survey the terminology used in English to describe these sets and to make proposals for standardizing the terminology. It is an extension of the author's earlier survey [**4**].

Many authors define function directly without reference to relations. Accordingly the survey has one section for authors who define functions in terms of relations and another section for those who introduce functions directly. There is also a brief reference to the terminology of category theory.

The choice of authors in the survey has been influenced by three factors: (1) the stature of the work in the literature, (2) a desire to adequately represent the variety of terminology in use, and (3) the availability of material in the libraries where the survey was performed.

2. Authors introducing functions by way of relations. None of the books consulted, except Bourbaki, gave a name to the set A. There was near-unanimity for the name of pr_1G and somewhat less agreement for the names of B and pr_2G. All the authors imposed Bourbaki's second condition for a relation to be a function, but few imposed the first condition. Very few of the authors considered have preserved Bourbaki's distinction between a function and its graph.

The following table does not present the full survey, but it is typical of the results.

Author	G	B	pr_1G	pr_2G	Conditions for function
Apostol [2]	relation	—	domain	range	(2)
Dieudonné [6]	graph	—	—	image	(2)
Halmos [8]	relation	—	domain	range	(1), (2)
Kelley [10]	relation	—	domain	range	(2)
Paley and Weichsel [17]	relation	codomain	domain	range or image	(2)
Suppes [18]	relation	—	domain	counter-domain, converse domain or range	(2)

3. Authors introducing functions directly. The authors in the survey who introduce functions directly without reference to a relation are almost unanimous in omitting any mention of the set A and talking only about pr_1G, which most of them call the *domain*. There is less agreement on names for the sets B and pr_2G, and still less agreement on what sort of entity a function is. Again, the following table does not present the complete survey, but it is representative of both the agreement and the diversity the survey revealed.

Author	Nature of Function	pr_1G	pr_2G	B
Allendoerfer and Oakley [1]	Relationship or set of ordered pairs	domain	range or set of values	—
Hu [9]	Rule	domain	image	range
Kleene [11]	Correspondence	range of independent variable or domain	range of dependent variable	—
Kuratowski [12]	Subset of Cartesian product	domain	image	range
Lefschetz [13]	Assignment	range	transform or image	—
MacLane [14]	Ordered triple of sets	domain	—	range

4. The terminology of category theory. There have been some attempts (for example Bush and Obreanu [5] and Gray [7]) to use the language of category theory to clarify the terminology used in discussing functions. If one follows MacLane [15] and Mitchell [16] we could call the set B the *codomain* of the relation or function. This is attractive if A is called the domain since the sets A and B are in a "correlationship." But our survey has shown that "domain" seldom refers to the set A.

5. Conclusions and proposals. The survey has shown that for many purposes it is not necessary to have names for all five of the sets that Bourbaki discusses. But examples can be given to show that each one does play a role in some part of the study of relations and functions. The distinction between a function that is *into* and one that is *onto* requires a distinction between the sets B and pr_2G. The problem of extending a given function is related to the distinction between A and pr_1G. Since all five sets play a role in the theory, it is useful to have names available for them when they are needed.

We have already indicated the disadvantages of the "coterminology" borrowed from category theory. It is with regret that we turn from this.

A literal translation of Bourbaki's terminology into English is somewhat cumbersome and inelegant.

An overwhelming majority of authors in English are in favor of calling the set pr_1G the *domain*. If we think of this as an abbreviation for "the domain of definition" we are close to the terminology of Bourbaki.

For the set pr_2G the vote is divided between *image* and *range*. Of these two, *image* seems to have this meaning more exclusively and on this ground is preferable.

Whether G is to be called the *graph of the relation* or the *relation* itself is more than a question of terminology. The question of whether a relation should be thought of as an ordered triple of sets or as a subset of a Cartesian product is beyond the scope of this survey.

We are left with the question of names for the sets A and B. Our survey has not provided any likely suggestions. We would like the names to be simple, brief English words that preserve the suggestive value of Bourbaki's French phrases. The names *source* for A and *target* for B seem to meet all these requirements. These names have no doubt been used informally by many mathematicians, but they do not appear to have gained the place in the literature that their usefulness deserves.

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GEOMETRICAL ASPECTS OF NEWTON'S METHOD

WALTER JENNINGS, Naval Postgraduate School, Monterey

Two of the most commonly used methods for solving systems of nonlinear equations are Newton's method and gradient methods including variations and combinations of these. The bibliography in Traub [1] lists some of the extensive literature of this complex subject. We shall discuss geometrical properties of these two methods for systems of two nonlinear equations and show that under certain circumstances they are one and the same.

The Newton-Raphson iteration

$$x_{k+1} = x_k - f(x_k)/f'(x_k)$$

for solving the nonlinear equation $f(x)=0$ is often referred to as the method of tangents since x_{k+1} is the x-intercept of the tangent at (x_k, y_k) to the curve $y=f(x)$. There is a similar geometrical interpretation of Newton's method for solving systems of nonlinear equations. We shall consider only the case of two equations

$$U(x, y) = 0, \qquad V(x, y) = 0. \tag{1}$$

The plane

$$z - U = (x - x_k)U_x + (y - y_k)U_y$$

is tangent to the surface $z = U(x, y)$ at the point $(x_k, y_k, U(x_k, y_k))$. The plane

$$z - V = (x - x_k)V_x + (y - y_k)V_y$$

is similarly related to the surface $z = V(x, y)$. The point of intersection of these two planes with the xy-plane is taken as the next approximation (x_{k+1}, y_{k+1}) to the solution of the system of nonlinear equations. The iteration equations for Newton's method are then

$$x_{k+1} = x_k - (UV_y - VU_y)/D \tag{2}$$
$$y_{k+1} = y_k - (VU_x - UV_x)/D$$

where

$$D = U_xV_y - V_xU_y,$$

all functions and their partial derivatives being evaluated at the point (x_k, y_k).

All solutions of the system (1) are zeros of the function

$$T(x, y) = \sqrt{U^2(x, y) + V^2(x, y)}$$

and conversely. The plane tangent to the surface $z=T(x, y)$ at the point $(x_k, y_k, T(x_k, y_k))$ cuts the xy-plane in the line

$$T_x(x_k, y_k)(x - x_k) + T_y(x_k, y_k)(y - y_k) + T(x_k, y_k) = 0. \tag{3}$$

By direct substitution one can verify that the point (x_{k+1}, y_{k+1}) defined by (2) lies in this line. Note that the line (3) is perpendicular to the gradient vector

$$T_x(x_k, y_k)\hat{\imath} + T_y(x_k, y_k)\hat{\jmath}.$$

The point $(\overset{*}{x}_{k+1}, \overset{*}{y}_{k+1})$ of intersection of the line (3) with a line through (x_k, y_k) parallel to the gradient vector is given by

$$\overset{*}{x}_{k+1} = x_k - \lambda T_x \tag{4}$$
$$\overset{*}{y}_{k+1} = y_k - \lambda T_y$$

where

$$\lambda = T/E, \qquad E = T_x^2 + T_y^2,$$

all functions being evaluated at point (x_k, y_k). Equations (4) give a particular gradient method, suggested by Cauchy in 1847 [2]. Note that $(\overset{*}{x}_{k+1}, \overset{*}{y}_{k+1})$ is the point which is nearest to (x_k, y_k) in the line of intersection of the xy-plane and the plane tangent to the surface $z=T(x, y)$. If we apply the Newton-Raphson method to the function

$$f(t) = T(x_k - tT_x(x_k, y_k), y_k - tT_y(x_k, y_k))$$

taking $t_0=0$ we find

$$t_1 = -f(0)/f'(0)$$
$$= T/(T_x^2 + T_y^2) = \lambda,$$

as given by Cauchy's gradient method.

The vectors

$$\boldsymbol{N} = (x_{k+1} - x_k)\hat{\imath} + (y_{k+1} - y_k)\hat{\jmath} = -\boldsymbol{N}_1/D$$

and

$$\boldsymbol{G} = (\overset{*}{x}_{k+1} - x_k)\hat{\imath} + (\overset{*}{y}_{k+1} - y_k)\hat{\jmath} = -\boldsymbol{G}_1/E$$

connect successive points defined by the two iterations. The vectors

$$N_1 = (UV_y - VU_y)\hat{\imath} + (VU_x - UV_x)\hat{\jmath}$$

and

$$G_1 = (UU_x + VV_x)\hat{\imath} + (UU_y + VV_y)\hat{\jmath}$$

are identical at any point not a solution of the system $U = V = 0$ only if

$$\begin{vmatrix} U_x - V_y & -U_y - V_x \\ U_y + V_x & U_x - V_y \end{vmatrix} = 0.$$

This latter equation is equivalent to the system of Cauchy-Riemann partial differential equations

$$U_x - V_y = 0, \ U_y + V_x = 0. \tag{5}$$

The vectors N_1 and G_1 are, of course, everywhere identical if $U+iV$ is an entire function. The vectors N and G may be thought to be 'bound' to the point (x_k, y_k). Since they terminate in the same line they will be identical whenever N_1 and G_1 are parallel. Hence *Newton's method is a gradient method if any one of the four functions* $\pm(U \pm iV)$ *is an entire function.* We can only conjecture that the converse is not true.

Figure 1 is a computer made graph illustrating Newton's method and Cauchy's gradient method. Let

$$U(x, y) = (x^2 \cos 2\theta + 2xy \sin 2\theta - y^2 \cos 2\theta - 1)/2, \qquad \theta = 65°,$$
$$V(x, y) = (x^2 - y^2 - 1)/2.$$

Here $U(x, y) = 0$ is the hyperbola $V(x, y) = 0$ rotated through 65° about the origin. The contour curves of the surface

$$z = T(x, y) = \sqrt{U^2(x, y) + V^2(x, y)}$$

were made using a FORTRAN program written by M. O. Dayhoff of the National Bio-Medical Research Foundation, Silver Springs, Maryland [3]. The function $T(x, y)$ was evaluated at points (x, y), $-3 \leqq x \leqq 3$, $-3 \leqq y \leqq 3$ at intervals on x and y of 0.05. These 14641 values were stored as a matrix. The Dayhoff program then generated lists of the x- and y-coördinates of successive points on contours at specified levels. A subroutine stored these on tape in a form suitable to drive the CalComp 565 plotter. In order to obtain relatively smooth curves the points on the curves through the iterates of both methods were adjusted so that the distance between successive points would not exceed 0.15 inches. The isolated point plots of the iterates were not so adjusted. The iterates given by Newton's method were plotted as squares and those given by Cauchy's gradient method as triangles. Initial points were taken along the boundary of the graph. The successive unadjusted iterates are connected by line segments. Some of these show a need to compare consecutive values of $T(x, y)$ to avoid 'overshooting.' If there is not a decrease the distance between con-

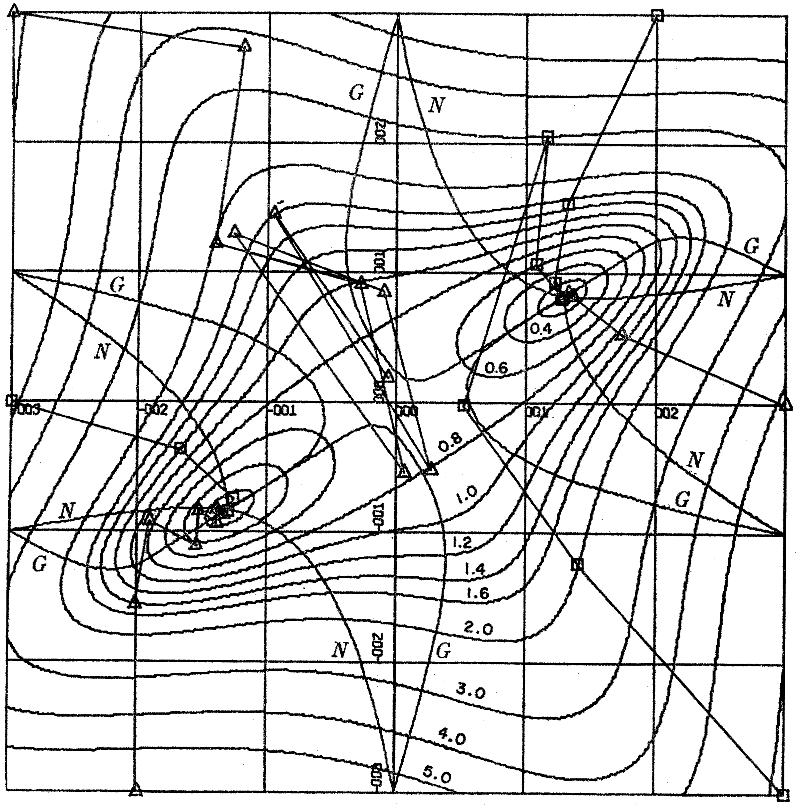

FIG. 1.

Newton's Method and Cauchy's Gradient Method for Solving $U(x, y)=0,\ V(x,y)=0$.
Contours of $T(x, y)=\sqrt{U^2+V^2}$.

secutive iterates should be shortened. In the case of the gradient method this will ensure convergence at least to a relative minimum of T. In the case of Newton's method one may occasionally have to shift to the direction of the gradient. Marquardt [4] has developed an algorithm based upon interpolating between the directions given by Newton's method and the negative gradient.

If $f(z)=U(x, y)+iV(x, y)$ is a complex function of the complex variable $z=x+iy$, its zeros can be found by solving the system $U(x, y)=0,\ V(x, y)=0$ by Newton's method. If $f(z)$ is an entire function the method will necessarily be a gradient method relative to the modulus function $|f(z)|$. Using the Cauchy-Riemann partial differential equations one can readily show that Newton's method applied to the system is equivalent to the Newton-Raphson iteration

$$z_{k+1} = z_k - f(z_k)f'(z_k)$$

for finding a complex zero of $f(z)$. Reference [5] suggests a modification of the Newton-Raphson method which exploits this observation.

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AN EXTENSION OF AN ELEMENTARY THEOREM IN CALCULUS

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Many forms of the classical theorem of L'Hôpital are used to evaluate so-called indeterminate forms 0/0 and ∞/∞. This rule is used for sequences as well as for functions. The purpose of this note is to give a theorem for sequences that is overlooked by every calculus text known to the author. The proof is also elementary and hence it is felt that because of its usefulness, it should be included in such texts.

THEOREM. *Let f, g be nonnegative continuous functions on* $[1, \infty)$. *Further let f be monotone decreasing and g be continuously differentiable on* $[1, \infty)$ *with* $\lim_{x\to\infty} g(x) = \infty$. *Then*

$$\lim_{n\to\infty} \frac{f(1) + f(2) + \cdots + f(n)}{g(n)} = \lim_{n\to\infty} \frac{f(n)}{g'(n)}$$

provided the latter limit exists.

Proof. If $S(n) = f(1) + f(2) + \cdots + f(n)$, then $S(n) - f(1) \leqq \int_1^n f(x)dx \leqq S(n)$, so that dividing by $g(n)$ and noting that $(f(1)/g(n)) \to 0$, we get

$$\lim_{n\to\infty} (S(n)/g(n)) = \lim_{n\to\infty} \left(\int_1^n f(x)dx\right)/g(n)$$

from which the result follows using the classical rule of L'Hôpital.

As an example we compute

$$\lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + 1/n\right) \Big/ \log n = \lim_{n\to\infty} \frac{1/n}{1/n} = 1.$$

The result probably holds for more general choices of f, but this would no longer be a problem in elementary calculus. For example, one computes $\lim_{n\to\infty} (1^p + 2^p + \cdots + n^p)/n^{p+1} = \int_0^1 x^p dx = 1/(p+1)$. This result, however, would follow immediately from the above theorem.

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted. Solutions should be submitted on separate, signed sheets. Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before February 1, 1970.

PROPOSALS

740. *Proposed by Dewey C. Duncan, Los Angeles, California.*

Can the decimal natural number consisting of $6k-1$ ones be a prime number?

741. *Proposed by Leon Bankoff, Los Angeles, California.*

A square and a triangle of equivalent area are inscribed in a semicircle, one side of the triangle forming the diameter of the semicircle. Show that the incenter of the triangle lies on one of the sides of the square.

742. *Proposed by S. Srinivasan, Panjab University, Chandigarh, India.*

Let $F(n)=(1/n)\sum_{q=1}^{n}\phi(q)/q$, $n\geqq 1$, ϕ the Euler phi function. Show that:
1) $F(n)>\frac{1}{2}$
2) $\lim_{n\to\infty} F(n)=6/\pi^2$.

743. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let P be an interior point of a regular tetrahedron, $T\equiv A_1A_2A_3A_4$, with $p_i=PA_i$, and let x_{ij} denote the distance of P from the edge A_iA_j. Then prove

$$\sum_{i=1}^{4} p_i \geqq 2\sqrt{3}/3 \sum_{i<j} x_{ij},$$

equality holding if and only if P is at the center O of T.

744. *Proposed by B. J. Cerimele, North Carolina State University.*

Show that the alternating series

$$\sum_{i=1}^{\infty} (-1)^{i+1}\ln(1+1/i)$$

is conditionally convergent and determine its sum.

745. *Proposed by Sidney H. L. Kung, Jacksonville University, Florida.*

Given any nine points in a unit square, show that among all the triangles having vertices on the given points there exists at least one triangle whose area does not exceed 1/8. Generalize this result.

746. *Proposed by Murray S. Klamkin, Ford Scientific Laboratory, and Morris Morduchow, Polytechnic Institute of Brooklyn.*

Determine the extreme values of $S_1/r+S_2/(n-r)$ where n is a fixed integer, $S_1=p_1+p_2+\cdots+p_r$,

$$S_1+S_2=\sum_{i=0}^{n-1} i,$$

and the p's are distinct integers in the interval $[0, n-1]$.

SOLUTIONS

Late Solutions

Joel Spencer, The RAND Corporation, Santa Monica, California: **714**; *K. L. Yocom, South Dakota State University*: **717**.

An Apt Alphametic

719. [March, 1969] *Proposed by J. A. H. Hunter, Toronto Canada.*

This apt alphametic was suggested by D. Murdoch. Simple mathematics can be fun when taught properly! So what must *DROP OUTS* represent?

$$\begin{array}{cccc} D & O & U & R \\ D & O & N & S \\ D & O & N' & T \\ S & T & O & P \\ D & R & O & P \\ \hline O & U & T & S \end{array}$$

Solution by Charles W. Trigg, San Diego, California.

In accordance with good practice in this problem in the decimal system, it is assumed that none of the numbers begins with zero. The letter O is replaced by θ.

From column one, $\theta \geqq 6$. Hence, at least 2 must carry from column two. Therefore $D=1$, $\theta=8$ or 9, $S=2$, and at least 2 must carry from column three. Then if $\theta=8$, $T+R=3$. But from column four, R and T are both odd or both even, so $\theta=9$, and $R+T<10$.

From column four, $R+T+2P=10$, 20 or 30. Hence possibilities are: $(R, T, P)=(0, 4, 3)$, $(0, 4, 8)$, $(0, 6, 7)$, $(0, 8, 6)$ and $(3, 5, 6)$, where values of R and T are interchangeable in the columns. The first triad leads to the unique solution:

$$1954 + 1982 + 1980 + 2093 + 1493 = 9502.$$

Also solved by Merrill Barnebey, Wisconsin State University at La Crosse; Richard L. Breisch, University of Colorado; Maxey Brooke, Sweeny, Texas; Martin J. Brown, University of Kentucky at Covington; Heather J. Bry, New York, New York; Ed and Betty Hatch, University of Santa Clara, California; Thomas Hughes, Arlington, Texas; David Isaacs, Valley Station, Kentucky; Alfred Kohler, Long Island University; Norton Levy, Concord-Carlisle High School, Massachusetts; George A. Novacky, St. Anselm High School, Swissvale, Pennsylvania; Thomas F. Parsons, Eastern Washington State College; Boyd A. Pearson and Dale Flowers, USN WL, Dahlgren, Virginia (jointly); Marlow Sholander, Case Western Reserve University; E. F. Schmeichel, Itasca, Illinois; E. P. Starke, Plainfield, New Jersey; Kenneth M. Wilke, Topeka, Kansas; Samuel Wolf, Linthicum Heights, Maryland; and the proposer. Four incorrect solutions were received.

A Shuffling Device

720. [March, 1969] *Proposed by Alfred Kohler, Long Island University.*

A device that shuffles cards always rearranges them in the same way relative to the order in which they are given to it. All of the hearts arranged in order from ace to king were put into the device, and then these shuffled cards were put into the device again to be shuffled a second time.

a. If the cards emerged in the order 10, 9, Q, 8, K, 3, 4, A, 5, J, 6, 2, 7, what order were the cards in after the first shuffle?

b. Show that it is impossible for the order of the cards after the second shuffle to be: Q, 8, 9, J, K, A, 4, 10, 5, 2, 7, 6, 3.

c. What is the minimum number of times that the device would have to shuffle thirteen cards before one could be certain that the cards had been restored at least once to their original order?

Solution by Mannis Charosh, Brooklyn, New York.

Using substitution notation we write the second shuffle as follows:

$$S^2 = \begin{pmatrix} A & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & J & Q & K \\ 10 & 9 & Q & 8 & K & 3 & 4 & A & 5 & J & 6 & 2 & 7 \end{pmatrix}$$
$$= A \quad 10 \quad J \quad 6 \quad 3 \quad Q \quad 2 \quad 9 \quad 5 \quad K \quad 7 \quad 4 \quad 8.$$

The problem now is to find S^1.

We take part (c) first. Since S^2 is cyclic, S^1 is also cyclic. Therefore S^{13} is the smallest positive power of S which equals I, the identity substitution. Translating, this means that 13 is the minimum number of shuffles required to restore the original order.

a. We have $(S^2)^7 = S^{14} =$

$$S^1 = \begin{pmatrix} A & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & J & Q & K \\ 9 & A & 4 & Q & J & 7 & 3 & 2 & 10 & 5 & K & 8 & 6 \end{pmatrix}.$$

b. We write the second shuffle as follows:

$$S^2 = \begin{pmatrix} A & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & J & Q & K \\ Q & 8 & 9 & J & K & A & 4 & 10 & 5 & 2 & 7 & 6 & 3 \end{pmatrix}$$
$$= (A \quad Q \quad 6)(2 \quad 8 \quad 10)(3 \quad 9 \quad 5 \quad K)(4 \quad J \quad 7).$$

This can be written as the product of $2+2+3+2$ or 9 transpositions. But this is impossible because the square of any substitution must form an even substitution. [G. Chrystal, *Algebra*, vol. II, p. 31, Cor. 4.] This proves part (b).

More generally, let n cards be given whose initial order is 1, 2, 3, $\cdots n$. Let the substitution representing the rth shuffle be the product of independent cycles $C_1, C_2, \cdots C_k$, which contain $c_1, c_2, \cdots c_k$ elements respectively. If $t=$ the LCM of the c's, then $S^t=I$. Let r be prime to t; then if $rx\equiv 1 \pmod t$, $(S^r)^x=S^1$.

If r is not prime to t, solutions for S^1 can be found (if they exist) by trial and error, working within each cycle.

Also solved by Richard L. Breisch, University of Colorado; Thomas J. Downs, Case Western Reserve University; Harry M. Gehman, SUNY at Buffalo, New York; Michael Goldberg, Washington, D.C.; John E. Homer, Union Carbide Corporation, Chicago, Illinois; Thomas Hughes; Bill Knight, University of Wyoming; David Isaacs, Valley Station, Kentucky; Norton Levy, Concord-Carlisle High School, Concord, Massachusetts; Michael J. Martino, Temple University; George A. Novacky, St. Anselm High School, Swissvale, Pennsylvania; Robert E. Pearson, Jr., Fort Mead, Maryland; Norman A. Robins, Illinois Institute of Technology; E. F. Schmeichel, College of Wooster, Ohio; E. P. Starke, Plainfield, New Jersey; Norman Sweet, SUNY at Oneonta, New York; Charles W. Trigg, San Diego, California; Kenneth M. Wilke, Topeka, Kansas; Samuel Wolf, Linthicum Heights, Maryland; and the proposer.

Segments Subtending Equal Angles

721. [March, 1969] *Proposed by Charles W. Trigg, San Diego, California.*

The eye of an observer standing in a ditch is level with the surrounding horizontal terrain. At a distance of x inches he observes a vertical pole on which there are three marks that divide the pole from the bottom into segments a, b, c, and d. If the four segments subtend equal angles at the eye of the observer, then:

1. Find x;
2. Determine limiting conditions on a and b;
3. Ascertain the smallest integer values of x, a, b, c, and d;
4. Compute the height of the pole;
5. Calculate the angle subtended by each segment.

Solution by Alfred Kohler, Long Island University.

Let segments a, b, c and d subtend angles at the eye of the observer of size α, β, γ and δ respectively and note that $\alpha=\beta=\gamma=\delta$. Now we have:

$\alpha=\text{arc tan } a/x$

$\beta=\text{arc tan } (a+b)/x-\text{arc tan } a/x$

$\gamma=\text{arc tan } (a+b+c)/x-\text{arc tan } (a+b)/x$

and

$\delta=\text{arc tan } (a+b+c+d)/x-\text{arc tan } (a+b+c)/x.$

If $u=\text{arc tan } x-\text{arc tan } y$ then $\tan u=(x-y)/(1-xy)$. This relationship applied three times yields:

$\beta=\text{arc tan } bx/(x^2-a^2+ab)=\text{arc tan } cx/[x^2+(a+b)(a+b+c)]$ and

$\delta=\text{arc tan } dx/[x^2+(a+b+c)(a+b+c+d)].$

Since $\tan\alpha=\tan\beta$, we have that $a/x=bx/(x^2+a^2+ab)$. Solving for x we obtain:

$$x = a\sqrt{(b+a)/(b-a)}. \tag{1}$$

Next, since $\tan\alpha=\tan\delta$ we have $a/x=cx/[x^2+(a+b)(a+b+c)]$. Substituting for x and solving for c we obtain:

$$c = b^2/(2a-b). \tag{2}$$

In a like manner we obtain:

$$d = ab^3/[(2a-b)(2a^2-b^2)]. \tag{3}$$

From (1) we see that $a<b$.
From (2) we see that $b<2a$.
From (3) we see that $b<a\sqrt{2}$.
Hence $a<b<a\sqrt{2}$.

Since $x=a\sqrt{(b+a)/(b-a)}$, for x to be an integer we must have the radicand of the form u^2/v^2 where u and v are positive integers such that $v\mid a$, $(u,v)=1$ and $u>v$. Solving for a and b we obtain:

$$a = k/2(u^2-v^2)$$
$$b = k/2(u^2+v^2)$$

where k is any positive integer. Solving for x, c and d we have:

$$2x = k(u/v)(u^2-v^2)$$
$$2c = k(u^2+v^2)^2/(u^2-3v^2)$$

and

$$2d = \frac{k(u^4-v^4)(u^2+v^2)^2}{(u^2-3v^2)(u^4-6u^2v^2+v^4)}.$$

Bearing in mind the restrictions that $a<b<a\sqrt{2}$ the smallest integer solutions are obtained by taking $u=3$, $v=1$ and $k=21$ which yield the values: $a=84$, $b=105$, $c=175$, $d=500$, and $x=252$. The height of the pole is $h=864$.

The angle α subtended by each segment is given by $\alpha=\arctan a/x$ which here becomes $\alpha=\arctan 84/252=\arctan\frac{1}{3}$ or $\alpha\doteq 18°26'$.

Also solved by E. P. Starke, Plainfield, New Jersey; Kenneth M. Wilke, Topeka, Kansas; and the proposer. One incorrect solution was received.

An Inequality

722. [March, 1969] *Proposed by Gregory Wulczyn, Bucknell University, Pennsylvania.*

Prove that $n(n+1)(2n+1)\geqq\dfrac{12}{n-1}\sum_{i,j=1}^{n} ij;\ i<j.$

I. *Solution by Kenneth A. Ribet, Brown University.*

We have

$$\sum_{i<j} ij = \frac{1}{2}\left[\left(\sum i\right)^2 - \sum i^2\right]$$

$$= \frac{1}{2}\left(\frac{n(n+1)}{2}\right)\left[\frac{n(n+1)}{2} - \frac{2n+1}{3}\right]$$

$$= \frac{1}{24} n(n+1)(3n^2+3n-4n-2)$$

$$= \frac{1}{24}(n)(n+1)(3n^3-n-2).$$

Accordingly, we have only to show that

$$3n^2 - n - 2 \leqq 2(2n+1)(n-1).$$

But this reduces to the inequality

$$0 \leqq n^2 + n + 1$$

which is obvious.

II. *Solution by L. Carlitz, Duke University.*

$$2 \sum_{1\leqq i<j\leqq n} ij = \sum_{i,j=1}^{n} ij - \sum_{i=1}^{n} i^2$$

$$= \frac{1}{4} n^2(n+1) - \frac{1}{6} n(n+1)(2n+1)$$

$$= \frac{1}{12} n(n+1)(n-1)(3n+2).$$

$$\frac{12}{n-1} \sum_{1\leqq i<j\leqq n} ij = \frac{1}{2} n(n+1)(3n+2) < n(n+1)(2n+1).$$

III. *Solution by Paul J. Zwier, Palo Alto, California.*
It is well known

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Thus, it is sufficient to show that

$$\sum_{i=1}^{n} i^2 \geqq \frac{2}{n-1} \sum_{i,j=1}^{n} ij \qquad (i<j).$$

The proof proceeds by induction on n. For $n=2$, the statement is obvious, since $5\geqq 4$. Assuming that

$$\sum_{i=1}^{n} i^2 \geqq \frac{2}{n-1} \sum_{i,j=1}^{n} ij,$$

we find that

$$* \qquad \sum_{i=1}^{n+1} i^2 \geqq \frac{2}{n-1} \sum_{i,j=1}^{n} ij + (n+1)^2 \geqq \frac{2}{n}\left[\sum_{i,j=1}^{n} ij + \frac{n}{2}(n+1)^2\right].$$

But $\frac{n}{2}(n+1)^2=(n+1)(1+2+3+\cdots+n)$. Hence the right side of * is

$$\frac{2}{n} \sum_{i,j=1}^{n+1} ij \qquad (i<j)$$

and the result follows by induction.

Also solved by Donald Batman, APO San Francisco; C. R. Berndtson, M.I.T. Lincoln Laboratory; J. Binz, Bern, Switzerland; Arthur R. Bolder, Brooklyn College; Richard L. Breisch, University of Colorado; Mickey Dargitz, Ferris State College, Michigan; Huseyin Demir, Middle East Technical University, Ankara, Turkey; Martin Fenerman, USN Oceanographic Office, Washington, D.C.; William F. Fox, Moberly Junior College, Missouri; Harry M. Gehman, SUNY at Buffalo; John E. Hafstrom, California State College at San Bernardino; Ignacio D. Herssein, Brooklyn, New York; John M. Howell, Los Angeles City College; Thomas Hughes, Donnelly J. Johnson, US Air Force Academy; Bruce W. King, Adirondack Community College, New York; Alfred Kohler, Long Island University; J. R. Kuttler, Johns Hopkins University, Applied Physics Laboratory; A. P. Lalchandani, Proctor and Gamble Company, Cincinnati, Ohio; Norton Levy, Concord-Carlisle High School, Massachusetts; Peter A. Lindstrom, Genesee Community College, New York; John J. Moore, Niagara University, New York; W. O. J. Moser, McGill University; Edward Moylan, Ford Motor Company, Dearborn, Michigan; Rachel H. Netzband, Cazenovia College, New York; Frank J. Papp, University of Delaware; C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsylvania; Willis B. Porter, New Iberia, Louisiana; Simeon Reich, Israel Institute of Technology, Haifa, Israel; Steve Rohde, Lehigh University; Nathan Rubinstein, Johns Hopkins University Applied Physics Laboratory; Edward F. Schmeichel, College of Wooster, Ohio; G. P. Speck, Bradley University; E. P. Starke, Plainfield, New Jersey; Michael Stolnicki, Oakland Community College, Michigan; Kenneth M. Wilke, Topeka, Kansas; K. Y. Yeung, Hong Kong; Kenneth L. Yocom, South Dakota State University; and the proposer.

An Ellipse and Its Astroid

723. [March, 1969] *Proposed by Stanley Rabinowitz, Far Rockaway, New York.*

Find the ratio of the major axis to the minor axis of an ellipse which has the same area as its evolute.

Solution by Rachel H. Netzband, Cazenovia College, New York.

Let a, b be respectively the major and minor axes of an ellipse and (x, y) its coordinates with parametric representation

$$x=a\cos t, \qquad y=b\sin t \quad (0\leqq t\leqq 2\pi).$$

Let (u, v) be a point on the evolute of the ellipse; then, as is known,

$$u = x - \frac{(1 + y'^2)y'}{y''}; \qquad v = y + \frac{1 + y'^2}{y''}.$$

One calculates easily

$$u = \frac{\cos^3 t}{a}(a^2 - b^2); \qquad v = -\frac{\sin^3 t}{b}(a^2 - b^2).$$

The area of the evolute is therefore

$$A_{\text{evol}} = 4\int_{t=0}^{\pi/2} v\,du = \frac{3\pi}{8}\,\frac{(a^2 - b^2)^2}{ab}.$$

By the theorem

$$\pi ab = \frac{3\pi}{8ab}(a^2 - b^2)^2,$$

$$8(ab)^2 = 3(a^2 - b^2)^2,$$

$$\sqrt{\tfrac{3}{2}}(a^2 - b^2) = 2ab,$$

$$\frac{a}{b} = \frac{\sqrt{6} + \sqrt{15}}{3} \approx 2.11.$$

Also solved by Leon Bankoff, Los Angeles, California; Michael Goldberg, Washington, D.C.; Lew Kowarski, Morgan State College, Maryland; C. B. A. Peck, Ordance Research Laboratory, State College, Pennsylvania; Simeon Reich, Israel Institute of Technology, Haifa, Israel; Nathan Rubinstein, Johns Hopkins Applied Physics Laboratory; E. P. Starke, Plainfield, New Jersey; and the proposer. Four incorrect solutions were received.

Triangle Probability

724. [March, 1969] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Find the probability that for a point P taken at random in the interior of a triangle ABC $(a \geqq b \geqq c)$, the distances of P from the sides of ABC form the lengths of sides of a triangle.

Solution by L. Carlitz, Duke University.

Let the internal angle bisectors meet the sides BC, CA, AB in L, M, N, respectively. Let x, y, z denote the distances from the point P to the sides BC, CA, AB. The equation $x = y + z$ represents a straight line, namely MN. Similarly $y = z + x$ represents NL, $z = x + y$ represents LM. The incenter I is in the interior of the triangle LMN; hence by continuity the point P must be restricted to the interior of LMN, so that the desired probability is equal to

$$p = \frac{\text{area } LMN}{\text{area } ABC} = \frac{\Delta_0}{\Delta}.$$

Now

$$CL = \frac{ab}{b+c}, \qquad CM = \frac{ab}{a+c},$$

so that

$$\text{area } CLM = \frac{1}{2}\,\frac{a^2b^2 \sin\gamma}{(a+c)(b+c)} = \frac{ab\Delta}{(a+c)(b+c)}.$$

By symmetry

$$\text{area } AMN = \frac{bc\Delta}{(b+a)(c+a)}, \qquad \text{area } BLN = \frac{ac\Delta}{(a+b)(c+b)}.$$

Thus

$$\begin{aligned}\Delta_0 = \text{area } LMN &= \Delta - \sum \frac{bc\Delta}{(b+a)(c+a)}\\ &= \Delta\left\{1 - \frac{\sum bc(b+c)}{(a+b)(b+c)(c+a)}\right\}\\ &= \Delta\,\frac{2abc}{(a+b)(b+c)(c+a)}\end{aligned}$$

so that

$$p = \frac{2abc}{(a+b)(b+c)(c+a)}.$$

Since $a+b \geqq 2\sqrt{ab}$, it follows that

$$p \leqq \tfrac{1}{4}$$

with equality only when $a=b=c$.

Also solved by Michael Goldberg, Washington, D.C.; C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsylvania; F. G. Schmitt, Jr., Berkeley, California (partially); Paul J. Zwier, Palo Alto, California; and the proposer.

Comment on Problem 533

533. [November, 1963, and May, 1964] *Proposed by David L. Silverman, Beverly Hills, California.*

What is the largest integer which cannot be partitioned into distinct squares?

Comment by Andrzej Makowski, Warsaw, Poland.

The result that every integer greater than 128 is the sum of different squares is due to R. Sprague in his paper, *Über Zerlegungen in ungleiche Quadratzahlen*, Mathematische Zeitschrift, 51 (1949) 289–290. See also the paper by H. and E. Richert, *Über Zerlegungen in paarweise verschiedene Zahlen*, Norsk Matematisk Tidsskrift, 31 (1949) 120–122. One can observe that the solution in this magazine is based upon the theorem of this paper.

Comment on Problem 601

601. [November, 1965, and May, 1966] *Proposed by David Singmaster, University of California, Berkeley.*

Let ϕ be Euler's function. It is well known that $\phi(x)=14$ has no solutions and that 14 is the smallest even number with this property. Show that there are infinitely many integers m such that the equation $\phi(x)=2m$ has no solutions.

Comment by Andrzej Makowski, Warsaw, Poland

This problem was proposed by M. G. Beumer in Elemente der Mathematik, 10 (1955) 22, Problem 230, and solved in vol. 11 (1956) 37. The more general results have been obtained by A. Schinzel, *Sur l'équation $\phi(x)=m$*, Elemente der Mathematik, 11 (1956) 75–78 and by P. Erdös, *Some remarks on Euler's ϕ-function*, Acta Arithmetica, 4 (1958) 10–19. Schinzel proved that for every integer n there exist infinitely many integers m divisible by n such that $\phi(x)=m$ has no solution.

Comment on Q418

Q418. [November, 1967] Given two primes separated by $2k-1$ integers with both primes greater than $2k+1$. What is the g.c.d. for the product of all the integers between two such primes?

[*Submitted by Brother Alfred Brousseau*]

Comment by Charles W. Trigg, San Diego, California.

Presumably what is meant is—Designate by P_i the product of the $2k-1$ integers which lie between two primes, each of which is greater than $2k+1$. What is the g.c.d. of all the P_i? The argument given in A418 goes only part of the way. Including the bounding primes there are $2k+1$ consecutive integers whose product consequently is divisible by $(2k+1)!$ Since both of these primes are greater than $2k+1$ they do not enter into this factorial, so P_i is divisible by $(2k+1)!$ which is the *common divisor*, but not necessarily the g.c.d., of all the P_i.

Consider the situation (13) (14) (15) (16) (17) and (19) (20) (21) (22) (23). Here $P_1=4(7)(5!)$ and $P_2=11(7)(5!)$. The g.c.d. of P_1 and P_2 is greater than 5! But for (37) (38) (39) (40) (41), $P_3=2(13)\ (19)\ (5!)$. So to prove the proposition we assume that the P_i are unlimited in number. Then if some P_i contains a prime factor $>2k+1$, since there are only $2k-1$ consecutive integers in P_i at least one other P_i must be devoid of that factor. Therefore $(2k+1)!$ is the g.c.d. of the P_i.

Comment on Q438

Q438. [September, 1968] Show that $(3n)!/(6^n n!)$ is a positive integer if n is.

[*Submitted by Charles W. Trigg*]

Comment by Lester Rubenfeld, Rensselaer Polytechnic Institute.

The number of even integers in the sequence 1, 2, 3, $\cdots$, m is $[\frac{m}{2}]$ (the greatest integer in $m/2$).

Consider the sequence 1, 2, $\cdots$, $3n$. The following numbers are clearly in the sequence: $3(n)$, $3(n-1)$, $3(n-2)$, $\cdots$, $3(1)$. The number of even numbers of the above form is $[\frac{n}{2}]$ (since $3k$ is even only when k is even). Thus, the number of even integers in the sequence 1, 2, $\cdots$, $3n$, *not* of the form $3k$, is

$$\left[\frac{3n}{2}\right] - \left[\frac{n}{2}\right] = n.$$

Thus, $(3n)!$ must have a factor of the form

$$3(n)\cdot 3(n-1)\cdots 3(1)\cdot 2^n = 6^n n!$$

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q463. Show that the vertex of a triangle, the point of contact of the excircle relative to this vertex with the opposite side, and the remote extremity of the diameter of the incircle perpendicular to the side are collinear.

[*Submitted by Charles W. Trigg*]

Q464. Find all the real solutions of the equation

$$\arcsin\left(\frac{x^2-8}{8}\right) = 2\arcsin\frac{x}{4} - \frac{\pi}{2}.$$

[*Submitted by Bryon L. McAllister*]

Q465. Prove that the product of any n consecutive positive integers is divisible by $n!$

[*Submitted by E. F. Schmeichel*]

Q466. If AB and BA are both identity matrices, then A and B are both square matrices.

[*Submitted by Murray S. Klamkin*]

Q467. Are there any persons alive today who can make the SQUARED claim: "I was x years old in the year x^2"?

[*Submitted by S. J. Farlon*]

(Answers on pages 243–244)

We now prove that $n-1 \in S$. For if not then $a^{n-1}/b \notin I$. Hence there exists a unique $A \in I$ such that

$$A < a^{n-1}/b < A + 1.$$

Since $a^n = Nb^n$, we have

$$a^n - Aab = Nb^n - Aab$$

from which we obtain

$$a/b = (Nb^{n-1} - Aa)/(a^{n-1} - Ab).$$

But $0 < a^{n-1} - Ab < b$ and this contradicts our choice for b. In particular the theorem has been proved for $n = 2$.

Hence S is not empty and therefore it must have a least element, say m. Suppose $m > 1$. Then $a^m/b \in I$ and $a^{m-1}/b \notin I$. Therefore there exists $K \in I$ such that $a^m = bK$ and there exists a unique $B \in I$ such that

$$B < a^{m-1}/b < B + 1.$$

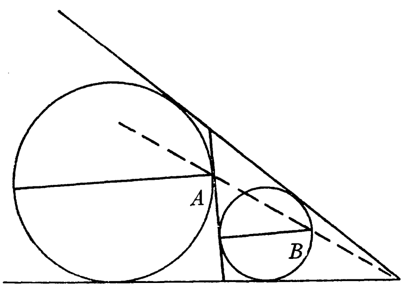
Hence $a^m - Bab = bK - Bab$, from which we obtain $a/b = (K - Ba)/(a^{m-1} - Bb)$; but $0 < a^{m-1} - Bb < b$ which again contradicts our choice for b. Hence $m = 1$ and the proof is complete.

Reference

1. E. A. Maier and Ivan Niven, A method of establishing certain irrationalities, this MAGAZINE, 37 (1964) 208–210.

ANSWERS

A463. The intersection of the common external tangents of two circles is the external center of similitude. A and B are homologous points. Homologous points are collinear with the center of similitude.



A464. The problem makes sense for $-4 \leqq x \leqq 4$. Since 0 and 4 are obviously solutions, we work with the intervals $A = (0, 4)$ and $B = (-4, 0)$. On A, the two sides of the equation differ by a constant c, since they have the same derivative. Setting $x = 2$, we find that $c = 0$, so that every x in A is also a solution.

To show that no x in B is a solution we need only note that $\sin^{-1}\{(x^2-8)/8\}$ is an even function while $2\sin^{-1}(x/4)$ is both odd and increasing, so that the fact that the graphs of the two sides coincide to the right of 0 shows that they can't even intersect to the left of 0. Thus the solution set is the interval $[0, 4]$.

A465. If k is a nonnegative integer, then

$$\frac{(k+1)(k+2)\cdots(k+n)}{n!} = \frac{(k+n)!}{k!n!} = \binom{k+n}{n}$$

which is an integer. See Uspensky, *Elementary Theory of Numbers*, Page 100.

A466. Let A be $m\times n$ and B be $n\times m$ with $m\geqq n$. The rank of AB is m. But since the rank of a product of two matrices cannot exceed the rank of either of the two matrices which is at most n, we have $m\leqq n$. Thus $m=n$. The result is also valid if AB and BA are both nonsingular scalar matrices, e.g., if ABC, CAB and BCA are all nonsingular scalar matrices, then they are all square matrices.

A467. If a = person's age at the time of the SQUARED claim, b = year the person was born, then in order to make the SQUARED claim it must be true that $a^2=b+a$ at some time during the person's lifetime. Clearly a person can make the SQUARED claim at most once in his lifetime. From the equation $b=a^2-a$, construct the table:

a	1	2	3 ···	43	44	45
b	0	2	6	1806	1892	1980

Thus it appears that only a person born in 1892 could be alive today and make the SQUARED claim. He would have been 44 years old in the year $44^2=1936$.

(Quickies on page 277)

COIN STRINGS

JAN M. GOMBERT, Princeton University

Most puzzlists have encountered the following coin problem: one is given three pennies (P) and two dimes (D), arranged $PDPDP$. One may take a penny and a dime which are adjacent and move them somewhere along the same line. The desired position is $PPPDD$. This can be achieved in four moves:

PDPDP

1. *PDP PD*
2. *P PDPD*
3. *PDPP D*
4. *PPPDD*

Elementary Geometry for College

CHARLES W. TRYON, Mount San Antonio College

A concise treatment of geometry for the one-semester college course, this textbook makes use of both classic terminology in geometry and the set concepts and algebraic terminology of modern mathematics. The author presents the equivalent of a full-year high school course, carefully condensed to concentrate on the interplay of algebra and geometry in modern proof. Theorems are often stated in both classical and more modern form. Some features of geometry not required in a brief course are included in an appendix: properties of real numbers used in algebra, the use of the truth table in interpreting logical relations, and a geometric interpretation of limits. The author provides carefully selected sets of exercises within each chapter and summary tests at the end of most chapters. With Answers to Exercises and Summary Tests. 290 pages. $7.95

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and PAUL W. HAGGARD, East Carolina University

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3. FREQUENCY OF ISSUE: January, March, May, September, and November
4. LOCATION OF KNOWN OFFICE OF PUBLICATION *(Street, city, county, state, ZIP code)*: 1225 Connecticut Avenue, N. W., Washington, DC
5. LOCATION OF THE HEADQUARTERS OR GENERAL BUSINESS OFFICES OF THE PUBLISHERS *(Not printers)*: Same
6. NAMES AND ADDRESSES OF PUBLISHER, EDITOR, AND MANAGING EDITOR

PUBLISHER *(Name and address)*: Mathematical Association of America, 1225 Connecticut Avenue, N. W., Washington, DC

EDITOR *(Name and address)*: Dr. Stephen A. Jennings, University of Victoria, Victoria, B. C., Canada

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2. TITLE OF PUBLICATION: MATHEMATICS MAGAZINE
3. FREQUENCY OF ISSUE: January, March, May, September, and November
4. LOCATION OF KNOWN OFFICE OF PUBLICATION *(Street, city, county, state, ZIP code)*: 1225 Connecticut Avenue, N. W., Washington, DC
5. LOCATION OF THE HEADQUARTERS OR GENERAL BUSINESS OFFICES OF THE PUBLISHERS *(Not printers)*: Same
6. NAMES AND ADDRESSES OF PUBLISHER, EDITOR, AND MANAGING EDITOR

PUBLISHER *(Name and address)*: Mathematical Association of America, 1225 Connecticut Avenue, N. W., Washington, DC

EDITOR *(Name and address)*: Dr. Stephen A. Jennings, University of Victoria, Victoria, B. C., Canada

MANAGING EDITOR *(Name and address)*: Dr. Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo, Buffalo, NY

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POD Form 3526 May 1966